602935

EXAMPLES

ON THE

INTEGRAL CALCULUS.

BY JAMES HANN,

MATHEMATICAL MASTER OF KING'S COLLEGE SCHOOL, LONDON

HONORARY MEMBER OF THE PHILOSOPHICAL SOCIETY OF NEWCASTLE-UPON-TYNE,
JOINT AUTHOR OF MECHANICS FOR PRACTICAL MEN,
AUTHOR OF THEORY OF BRIDGES, JOINT AUTHOR OF NAUTICAL TABLES,
AUTHOR OF A TREATISE ON THE STEAM ENGINE,
AUTHOR OF THEORETICAL AND PRACTICAL MECHANICS,
AND AUTHOR OF PLANE AND SPHERICAL
TRIGONOMETRY.

PREFACE.

As this work contains a great number of Integrals fully worked out, the Author hopes that it will considerably facilitate the progress of those who are entering on this branch of study, by showing them almost all the artifices that are used in those branches that come within its scope.

The works that have been consulted are those of Peacock, Gregory, Hall, De Morgan, Young, and various mathematical periodicals; also the excellent little work on the Calculus by Mr. Tate, which, like all the productions of that eminent writer, abounds with useful information, apart from the able manner in which he has treated the first principles.

Where integration by parts is used, the whole process is put down, but the student should endeavour as soon as possible to acquire the facility of running off the quantities without writing down all the intermediate steps.

EXAMPLES

INTEGRAL CALCULUS.

CHAPTER I.

ELEMENTARY INTEGRALS TO BE COMMITTED TO MEMORY.

(1.) $\int \dfrac{dx}{x} = \log x.$

(2.) $\int \dfrac{dx}{a + b\,x^2} = \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(x \sqrt{\dfrac{b}{a}} \right)$, or $\int \dfrac{dx}{a^2 + x^2}$
$= \dfrac{1}{a} \tan^{-1} \dfrac{x}{a}.$

(3.) $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \dfrac{x}{a}.$

(4.) $\int \dfrac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1} \dfrac{x}{a}.$

(5.) $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \log \dfrac{(x + \sqrt{x^2 \pm a^2})}{a}.$

(6.) $\int \dfrac{dx}{\sqrt{2\,a\,x - x^2}} = \mathrm{vers}^{-1} \dfrac{x}{a}.$

(7.) $\int \dfrac{dx}{\sqrt{x^2 \pm 2\,ax}} = \log (x \pm a + \sqrt{x^2 \pm 2\,ax}.$

(8.) $\int \dfrac{dx}{x \sqrt{x^2 - a^2}} = \dfrac{1}{a} \sec^{-1} \dfrac{x}{a}.$

B

(9.) $\int \dfrac{d\,x}{x\sqrt{a^2 \pm x^2}} = \dfrac{1}{a}\log\dfrac{x}{a + \sqrt{a^2 \pm x^2}}.$

(10.) $\int \dfrac{a\,d\,x}{x^2 - a^2} = \dfrac{1}{2}\log\left(\dfrac{x-a}{x+a}\right).$

(11.) $\int a^x dx = \dfrac{a^x}{\log a}.$

(12.) $\int e^{ax}\,dx = \dfrac{e^{ax}}{a}.$

(13.) $\int \dfrac{'d\,\theta}{\sin\theta} = \log\tan\dfrac{\theta}{2}$

(14.) $\int \dfrac{d\,\theta}{\cos\theta} = \log\tan\left\{\dfrac{\pi}{4} + \dfrac{\theta}{2}\right\}$

(15.) $\int \dfrac{d\,\theta}{\sin\theta\cos\theta} = \log\tan\theta.$

Examples.

(1.) $\int \dfrac{d\,x}{1 + 5\,x^2} = \dfrac{1}{\sqrt{5}}\tan^{-1}(x\sqrt{5}).$

(2.) $\int \dfrac{d\,x}{x^2(a + bx)}.$ Let $x = \dfrac{1}{z}$, then the integral is reduced to $\int -\dfrac{z\,dz}{az + b} = -\dfrac{z}{a} + \dfrac{b}{a^2}\log(az + b)$

$$= -\dfrac{1}{ax} + \dfrac{b}{a^2}\log\left(\dfrac{a + bx}{x}\right).$$

(3.) $\int\left\{\dfrac{a\,d\,x}{x} + \dfrac{b\,d\,x}{x^2} + \dfrac{c\,dx}{x^3} + \dfrac{e\,dx}{x^4}\right\} = a\log x - \dfrac{b}{x}$

$$- \dfrac{c}{2\,x^2} - \dfrac{e}{3\,x^3}$$

(4.) $\int (1 + x^2)(1 + x)x\,d\,x = \int(1 + x + x^2 + x^3)\,x\,d\,x$

$$= \int (x + x^2 + x^3 + x^4)\, dx = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}.$$

(5.) $\displaystyle\int \frac{(1+x)^2(1-x)\,dx}{x^2} = \int \frac{1+x-x^2-x^3\,dx}{x^2} = \int \left(\frac{1}{x^2} + \frac{1}{x} \right.$

$\displaystyle\left. -1-x \right)dx = \log x - x - \frac{x^2}{2} - \frac{1}{x} = \frac{x \log x^5 - 2\,x^2 - x^3 - 2}{2\,x}$

(6.) $\displaystyle\int \frac{x^4\,d\,x}{x^2+1} = \int \left\{ x^2 - 1 + \frac{1}{x^2+1} \right\} d\,x = \frac{x^3}{3} - x + \tan^{-1} x.$

(7.) $\displaystyle\int du = \int \frac{x^3\,d\,x}{(x+2)^2}.$ Let $x + 2 = z;\ d\,x = dz$

$$\therefore\ du = \frac{\{z-2\}^3 dz}{(z)^2} = \frac{(z^3 - 6\,z^2 + 12\,z - 8)\,dz}{z^2}$$

$$= \left(z - 6 + \frac{12}{z} - \frac{8}{z^2} \right) d\,z,\ \therefore\ u = \frac{z^2}{2} - 6\,z + 12\,.\log z + \frac{8}{z},$$

$$= \frac{z^3 - 12\,z^2 + z\,.\log\,(z^{24}) + 16}{2\,z}$$

$$= \frac{(x+2)^3 - 12\,(x+2)^2 + (x+2)\log\,(x-2)^{24} + 16}{2\,(x+2)}.$$

(8.) $\displaystyle\int \frac{x^3\,d\,x}{x^2+1} = \int \left(x^3 - x + \frac{x}{x^2+1} \right) d\,x$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \log\,(\sqrt{x^2+1}).$$

(9.) $\displaystyle\int \frac{5\,d\,x}{2\,x^4 + 3\,x^2}.$ Let $x = \frac{1}{z}$, then the integral is

reduced to $\displaystyle\int \frac{-5\,z^2\,dz}{2 + 3\,z^2}$

$$= -\frac{1}{3} \int \frac{15\,z^2\,d\,z}{2 + 3\,z^2} \quad = -\frac{1}{3} \int \left(5\,d\,z - \frac{10\,d\,z}{3\,z^2+2} \right)$$

$$= \int \left(\frac{10}{3} \cdot \frac{d\,z}{3\,z^2+2} - \frac{5\,d\,z}{3} \right)$$

$$= \frac{1}{3} \left\{ \left(\frac{5\sqrt{2}}{\sqrt{3}} \right) \tan^{-1} \left(\sqrt{\frac{3}{2}} \cdot z \right) - 5\,z \right\}$$

$$= \frac{1}{3} \left(5\sqrt{\frac{2}{3}} \tan^{-1} \left(\sqrt{\frac{3}{2\,x^2}} \right) - \frac{5}{x} \right).$$

(10.) $\int \dfrac{x\sqrt{x}\,dx}{1+\sqrt{x}} = \int \left(x - \sqrt{x} + 1 - \dfrac{1}{1+\sqrt{x}} \right) d\,x$

$$= \frac{x^2}{2} - \frac{2\,x^{\frac{3}{2}}}{3} + x - \int \frac{d\,x}{1+\sqrt{x}}.$$

And $\int \dfrac{d\,x}{1+\sqrt{x}} =$ (by substituting $\sqrt{x}=z$)

$$2 \int \frac{z\,dz}{1+z} = 2 \int \left(d\,z - \frac{d\,z}{1+z} \right),$$

$$= 2\,z - \log\,(1+z)^2 = 2\sqrt{x} - \log\,(1+\sqrt{x})^2\,;$$

$\therefore$ the integral is $= \dfrac{x^2}{2} - \dfrac{2\,x^{\frac{3}{2}}}{3} + x - 2\,(x)^{\frac{1}{2}} + \log\,(1+\sqrt{})^2.$

(11.) $\int \dfrac{x^2\sqrt{x}\,.\,d\,x}{1+x}.$ Let $\sqrt{x}=z$, then $x=z^2$

$$d\,x = 2\,z\,d\,z\,;$$

$$\therefore \int \frac{x^2\sqrt{x}\,.\,d\,x}{1+x} = \int \frac{z^5}{1+z^2}\,.\,2\,z\,d\,z = 2 \int \frac{z^6\,d\,z}{1+z^2}$$

$$= 2 \int \left\{ z^4\,d\,z - z^2\,d\,z + d\,z - \frac{d\,z}{1+z^2} \right\}$$

$$+ \frac{2\,z^5}{5} - \frac{2\,z^3}{3} + 2\,z - 2\,\tan^{-1}z$$

$$= \frac{2}{5}\,x^{\frac{5}{2}} - \frac{2}{3}\,x^{\frac{3}{2}} + 2\,x^{\frac{1}{2}} - 2\,\tan^{-1}x^{\frac{1}{2}}$$

$$(12.) \int \frac{dx}{1+x+x^2} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{2x+1}{\sqrt{3}}\right\} \text{ (an useful integral)}$$

$$\int \frac{dx}{a+bx+cx^2} = \frac{1}{c}\int \frac{dx}{x^2+\frac{b}{c}x+\frac{a}{c}} = 2\int \frac{d(2cx+b)}{(2cx+b)^2+(4ac-b^2)}$$

$$= \frac{2}{\sqrt{4ac-b^2}} \tan^{-1}\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right).$$

Examples for Practice.

(1.) $\int (a+bx)dx = \dfrac{(a+bx)^2}{2b}$.

(2.) $\int x(a+bx^2)^3 dx = \dfrac{(a+bx^2)^4}{8b}$.

(3.) $\int \dfrac{(a+bx^2+cx^3)(3cx^2+2bx)dx}{4} = \dfrac{(a+bx^2+cx^3)^2}{8}$

(4.) $\int \dfrac{\{(x+\sqrt{x})(2\sqrt{x}+1)\}dx}{\sqrt{x}} = (x+\sqrt{x})^2$.

(5.) $\int_b^o (a+bx)xdx = \dfrac{3ab^2+2b^4}{6}$.

(6.) $\int \dfrac{(\sqrt{x^2+4}+x)dx}{\sqrt{x^2+4}} = (x+\sqrt{x^2+4})$.

(7.) $\int \dfrac{(\sqrt{x^2+4}+x)^2 dx}{\sqrt{x^2+4}} = \dfrac{1}{2}(x+\sqrt{x^2+4})^2$.

(8.) $\int (a+bx^2)^2 \cdot 2xb\, dx = \dfrac{(a+bx^2)^3}{3}$.

(9.) $\int \dfrac{3\,dx}{\sqrt{x^2+3}\,.\,(\sqrt{x^2+3}-x)} = x + \sqrt{x^4+3}.$

(10.) $\int \dfrac{(2x+1)\,dx}{\sqrt{x^2+x}} = \dfrac{\sqrt{x^2+x}}{\frac{1}{2}} = 2\sqrt{x^2+x}.$

(11.) $\int \dfrac{(3x+2)\,dx}{\sqrt{x+1}} = 2\sqrt{x^4+x^2}.$

(12.) $\int \dfrac{2x\,dx}{(x^2+1)^2} = -\dfrac{1}{x^2+1}.$

(13.) $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\log\sqrt{a+bx}.$

(14.) $\int \dfrac{(2x+1)\,dx}{\sqrt{x^2+x+1}} = 2\sqrt{x^2+x+1}.$

(15.) $\int \dfrac{\left(x^3+\dfrac{3c}{4d}x^2+\dfrac{b}{2d}x\right)dx}{\sqrt{a+bx^2+cx^3+dx^4}} = \dfrac{\sqrt{a+bx^2+cx^3+dx^4}}{2d}.$

(16.) $\int \dfrac{(9x^8+8x^7)\,dx}{(x^9+x^8)^{\frac{3}{5}}} = \dfrac{5\,.\,(x^9+x^8)^{\frac{2}{5}}}{2}.$

(17.) $\int \dfrac{\left(x^{n-1}+\dfrac{n-1}{n}x^{n-2}\right)dx}{(x^n+x^{n-1})^{\frac{p}{q}}} = \dfrac{q\,.\,(x^n+x^{n-1})^{\frac{q-p}{q}}}{n\,.\,(q-p)}.$

(18.) $\int \dfrac{\left(x+\dfrac{b}{2c}\right)dx}{\sqrt{a+bx+cx^2}} = \dfrac{\sqrt{a+bx+cx^2}}{c}.$

(19.) $\int \dfrac{\left(x+\dfrac{b}{2c}\right)dx}{(a+bx+cx^2)^{\frac{m}{n}}} = \dfrac{n\,.\,(a+bx+cx^2)^{\frac{n-m}{n}}}{(n-m)\,.\,2c}.$

CHAPTER II.

RATIONAL FRACTIONS.

(1.) To integrate $du = \dfrac{x\,dx}{(x+2)(x+3)^2}$.

Let $\dfrac{x}{(x+2)(x+3)^2} = \dfrac{A}{(x+3)^2} + \dfrac{B}{(x+3)} + \dfrac{P}{(x+2)}$,

$\therefore x = A(x+2) + B(x+3)(x+2) + P(x+3)^2$.

Let $x = -3$,

$$\therefore -3 = A(2-3) = -A, \quad \therefore A = 3,$$

$$x - 3(x+2) = B(x+3)(x+2) + P(x+3)^2$$

$$-2(x+3) = B(x+3)(x+2) + P(x+3)^2$$

$$\therefore -2 = B(x+2) + P(x+3).$$

Let $x = -3$,

$$-2 = B(2-3) = -B, \quad \therefore B = 2.$$

Let $x = -2$,

$$-2 = P(3-2) = P, \quad \therefore P = -2,$$

$$\therefore \frac{x}{(x+2)(x+3)^2} = \frac{3}{(x+3)^2} + \frac{2}{(x+3)} - \frac{2}{(x+2)},$$

$$\therefore \int \frac{x\,dx}{(x+2)(x+3)^2}$$

$$= \int \frac{3\,dx}{(x+3)^2} + 2\int \frac{dx}{(x+3)} - 2\int \frac{dx}{(x+2)}$$

$$= -\frac{3}{(x+3)} + \log\left(\frac{x+3}{x+2}\right)^2.$$

(2.) To find $\int \dfrac{x^2 dx}{(x+2)^2(x+4)^2}$.   Let

$$\frac{x^2}{(x+4)^2(x+2)^2} = \frac{A}{(x+4)^2} + \frac{B}{(x+4)} + \frac{C}{(x+2)^2} + \frac{D}{x+}$$

$$x^2 = A(x+2)^2 + B(x+4)(x+2)^2 + C(x+4)^2 +$$

$$D(x+2)(x+4)^2.$$

Let $x+2 = 0$, $\therefore x = -2$, $x^2 = 4$, and $C(x+4)^2 = 4C$

$$\therefore 4 = 4C, \quad \therefore C = 1.$$

Let $x+4 = 0$, $\therefore x = -4$, $x^2 = 16$, and $A(x+2)^2 = 4A$

$$\therefore A = 4,$$

$$\therefore x^2 - (x+4)^2 - 4(x+2)^2 = -4\{x^2 + 6x + 8\}$$

$$= -4\{(x+2)(x+4)\},$$

$$= B(x+4)(x+2)^2 + D(x+2)(x+4)^2,$$

$$\therefore -4 = B(x+2) + D(x+4).$$

Let $x = -2$, $\therefore -4 = 2D$, $\therefore D = -2$.

Let $x = -4$, $\therefore -4 = -2B$, $\therefore B = 2$.

The fraction reduced becomes, therefore,

$$\frac{4}{(x+4)^2} + \frac{2}{(x+4)} + \frac{1}{(x+2)^2} - \frac{2}{(x+2)};$$

and its integral is, therefore,

$$\int \frac{4\,dx}{(x+4)^2} + \int \frac{dx}{(x+2)^2} + \log\left(\frac{x+4}{x+2}\right)^2.$$

$$\int \frac{4\,dx}{(x+4)^2} = \int 4\,dx\,(x+4)^{-2};$$

$$= \frac{-4}{(x+4)}.$$

In the same way,

$$\int \frac{dx}{(x+2)^2} = \frac{-1}{x+2},$$

and $- \left\{ \frac{4}{x+4} + \frac{1}{x+2} \right\} = - \frac{5x+12}{x^2+6x+8};$

therefore the complete integral is

$$- \frac{5x+12}{x^2+6x+8} + \log \left(\frac{x+4}{x+2} \right)^2$$

(3.) $\int \frac{2x\,dx}{(x^2+1)(x^2+3)}.$

Let $\frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax}{x^2+1} + \frac{Bx}{x^2+3};$

$\therefore 2 = A(x^2+3) + B(x^2+1), \quad \therefore A+B=0,$

$3A+B=2, \quad \therefore A=1, B=-1,$

$$\int \frac{2x\,dx}{(x^2+1)(x^2+3)} = \int \left(\frac{x\,dx}{x^2+1} - \frac{x\,dx}{x^2+3} \right)$$

$$= \log \sqrt{\frac{x^2+1}{x^2+3}}$$

(4.) $du = \frac{x^2 dx}{(x^2+1)(x^2+4)}.$

Let $\dfrac{x^2}{(x^2+1)(x^2+4)} = \dfrac{A}{(x^2+1)} + \dfrac{B}{(x^2+4)}$

$$x^2 = A(x^2+4) + B(x^2+1).$$

Let $x = \sqrt{-1}$, or $x^2 = -1$,

$$\therefore \; -1 = 3A, \quad \therefore \; A = -\frac{1}{3},$$

$$\therefore \; x^2 + \frac{1}{3}(x^2+4) = B(x^2+1),$$

$$\therefore \; \frac{4(x^2+1)}{3} = B(x^2+1), \quad \therefore \; B = \frac{4}{3}$$

$$\int \frac{x^2\,dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int\left\{\frac{dx}{(x^2+1)} - \int\frac{4\,dx}{(x^2+4)}\right\}$$

$$= \frac{1}{3}\int\left\{\frac{dx}{(1+x^2)} - \frac{dx}{\left(1+\dfrac{x^2}{4}\right)}\right\}$$

$$= \frac{1}{3}\left\{2\tan^{-1}\frac{x}{2} - \tan^{-1}x\right\}.$$

Or thus, $\displaystyle\int \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\int \frac{3x^2\,dx}{(x^2+1)(x^2+4)}$

$$= \frac{1}{3}\int \frac{\{4x^2+4-(x^2+4)\}\,dx}{(x^2+1)(x^2+4)}$$

$$= \frac{1}{3}\int\left(\frac{4\,dx}{(x^2+4)} - \frac{dx}{x^2+1}\right)$$

$$= \frac{1}{3}\left(2\tan^{-1}\frac{x}{2} - \tan^{-1}x\right)$$

(5.) $du = \dfrac{(3x^2+x-2)\,dx}{(x-1)^3(x^2+1)}.$

Let $\dfrac{(3x^2 + x - 2)}{(x-1)^3 (x^2+1)}$

$$= \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)} + \frac{P}{(x^2+1)}$$

$$3x^2 + x - 2 = A(x^2+1) + B(x-1)(x^2+1) + C(x-1)^2(x^2+1)$$
$$+ P(x-1)^3.$$

Let $x = 1$ $\therefore 2 = 2A,$ $\therefore A = 1,$

$$3x^2 + x - 2 - (x^2+1) = B(x-1)(x^2+1) + C(x-1)^2(x^2+1)$$
$$+ P(x-1)^3,$$

$$\therefore (2x+3)(x-1) = B(x-1)(x^2+1) + C(x-1)^2(x^2+1)$$
$$+ P(x-1)^3,$$

$$\therefore (2x+3) = B(x^2+1) + C(x-1)(x^2+1) + P(x-1)^2.$$

Let $x = 1,$

$$\therefore 5 = 2B, \qquad \therefore B = \frac{5}{2},$$

$$2x + 3 - \frac{5}{2}(x^2+1)$$

$$= -\frac{5x^2 - 4x - 1}{2}$$

$$= -\frac{(5x+1)(x-1)}{2} = C(x^2+1)(x-1) + P(x-1)^3$$

$$-\frac{5x+1}{2} = C(x^2+1) + P(x-1); \text{ if } x = 1,$$

$$\therefore -3 = 2C, \qquad \therefore C = -\frac{3}{2}$$

$$-\frac{5x+1}{2} + \frac{3}{2}(x^2+1) = P(x-1)$$

$$\frac{3x^2 - 5x + 2}{2} = P(x-1)$$

$$\frac{(3x - 2)(x - 1)}{2} = P(x - 1),$$

$$\therefore P = \frac{3x - 2}{2} = \left(\frac{3x}{2} - 1\right)$$

$$\therefore \int \frac{(3x^2 + x - 2)\,dx}{(x - 1)^3(x^2 + 1)} = \int \frac{dx}{(x - 1)^3} + \frac{5}{2}\int \frac{dx}{(x - 1)^2}$$

$$- \frac{3}{2}\int \frac{dx}{(x - 1)} + \frac{3}{2}\int \frac{x\,dx}{(x^2 + 1)}$$

$$- \int \frac{dx}{(x^2 + 1)}$$

$$= -\frac{1}{2(x - 1)^2} - \frac{5}{2}\frac{1}{(x - 1)} - \frac{3}{2}\log(x - 1)$$

$$+ \frac{3}{2}\cdot\frac{1}{2}\int \frac{2x}{(x^2 + 1)} - \tan^{-1}x$$

$$= -\frac{1}{2(x - 1)^2} - \frac{5}{2}\cdot\frac{1}{x - 1} + \frac{3}{2}\log\frac{\sqrt{x^2 + 1}}{(x - 1)} - \tan^{-1}x$$

(6.) $du = \dfrac{(1 - x + x^2)\,dx}{1 + x + x^2 + x^3} = \dfrac{(1 - x + x^2)\,dx}{(1 + x)(1 + x^2)}.$

Let $\dfrac{1 - x + x^2}{(1 + x)(1 + x^2)} = \dfrac{A}{(1 + x)} + \dfrac{B}{(1 + x^2)},$

$$\therefore 1 - x + x^2 = A(1 + x^2) + B(1 + x)$$

$$x = -1,$$

then $3 = 2A,$ $\therefore A = \dfrac{3}{2},$

$$1 - x + x^2 - \frac{3}{2}(1 + x^2) = B(1 + x)$$

$$= -\frac{x^2 + 2x + 1}{2} = -\frac{(1 + x)^2}{2} = B(1 + x),$$

$$\therefore \mathrm{B} = -\frac{1+x}{2}$$

$$= -\frac{1}{2} - \frac{x}{2},$$

$$\therefore \int \frac{(1 - x + x^2)\, dx}{1 + x + x^2 + x^3}$$

$$= \int \left(\frac{3}{2} \cdot \frac{1}{(1+x)} - \frac{1}{2} \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot \frac{x}{1+x^2} \right) dx$$

$$= \frac{1}{2} \left(3 \log (1+x) - \tan^{-1} x - \frac{1}{2} \log (1+x^2)^{\frac{1}{2}} \right)$$

$$= \frac{1}{2} \left(\log \frac{(1+x)^3}{\sqrt[4]{1+x^2}} - \tan^{-1} x \right)$$

(7.) $du = \dfrac{dx}{x\,(1+x)^2\,(1+x+x^2)}.$

Let $\dfrac{1}{x\,(x+1)^2\,(1+x+x^2)}$

$$= \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{(1+x)^2} + \frac{\mathrm{C}}{(1+x)} + \frac{\mathrm{P}}{1+x+x^2},$$

$$\therefore 1 = \mathrm{A}\,(1+x)^2\,(1+x+x^2) + \mathrm{B}\,(1+x+x^2)\,x$$

$$+\ \mathrm{C}\,(1+x)\,(1+x+x^2)\,x + \mathrm{P}\,(1+x)^2\,x.$$

Let $x = 0,$ $\quad \therefore \mathrm{A} = 1,$

$$1 - (1+x)^2\,(1+x+x^2)$$

$$= \mathrm{B}\,(1+x+x^2)\,x + \mathrm{C}\,(1+x)\,(1+x+x^2)\,x + \mathrm{P}\,(1+x)^2\,x$$

$$\therefore\ -(3 + 4x + 3x^2 + x^3) = \mathrm{B}\,(1+x+x^2)$$

$$+\ \mathrm{C}\,(1+x)\,(1+x+x^2) + \mathrm{P}\,(1+x)^2.$$

Let $x = -1$, $\therefore$ B $= -1$,

$$\therefore \; -(3 + 4x + 3x^2 + x^3) + (1 + x + x^2) =$$

$$\text{C}(1 + x)(1 + x + x^2) + \text{P}(1 + x)^2 \text{ or}$$

$$-(2 + 3x + 2x^2 + x^3) = \text{C}(1 + x)(1 + x + x^2) + \text{P}(1 + x)^2$$

$$-(2 + x + x^2)(1 + x) = \text{C}(1 + x)(1 + x + x^2) + \text{P}(1 + x)^2$$

$$-(2 + x + x^2) = \text{C}(1 + x + x^2) + \text{P}(1 + x)$$

$$x = -1, \qquad \therefore \text{C} = -2,$$

$$-(2 + x + x^2) + 2(1 + x + x^2) = \text{P}(1 + x)$$

$$x(1 + x) = \text{P}(1 + x), \qquad \therefore \text{P} = x,$$

$$\therefore \int \frac{d x}{x(1 + x)^2(1 + x + x^2)} =$$

$$\int \left(\frac{dx}{x} - \frac{dx}{(1 + x)^2} - \frac{2\,dx}{(1 + x)} + \frac{x\,dx}{1 + x + x^2} \right)$$

$$= \frac{1}{(1 + x)} - 2\log(1 + x) + \int \frac{-(1 + x + 2x^2)\,dx}{x + x^2 + x^3}$$

$$= \frac{1}{1 + x} - 2\log(1 + x) +$$

$$\frac{1}{2}\int d x \left(\frac{2x + 3x^2 + 4x^3}{x^2 + x^3 + x^4} - \frac{x^2}{x^2 + x^3 + x^4} \right)$$

$$= \frac{1}{1 + x} + \log \frac{\sqrt{x^2 + x^3 + x^4}}{(1 + x)^2} - \int \frac{x^2\,dx}{x^2 + x^3 + x^4} \cdot \frac{1}{2}$$

$$\int \frac{x^2\, dx}{2\,(x^2 + x^3 + x^4)} = \int \frac{2\, dx}{4 + 4\,x + 4\,x^2} = \int \frac{2\, dx}{3 + (2\,x + 1)^2}$$

$$= \frac{2}{3} \int \frac{dx}{1 + \dfrac{(2\,x + 1)^2}{3}}; \ \text{let } \frac{2\,x + 1}{\sqrt{3}} = z,$$

$$\therefore\ dx = \frac{\sqrt{3}\,.\,dz}{2},$$

$$\therefore \frac{2}{3} \int \frac{dx}{1 + \dfrac{(2\,x - 1)^2}{3}} = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot \int \frac{dz}{1 + z^2} = \frac{1}{\sqrt{3}} \cdot \int \frac{dz}{1 + z^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} z = \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\,x + 1}{\sqrt{3}}\,.$$

$$\therefore \int \frac{dx}{x\,(1 + x)^2\,(1 + x + x^2)} = \frac{1}{1 + x} + \log \frac{\sqrt{x^2 + x^3 + x^4}}{(1 + x)^2}$$

$$- \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\,x + 1}{\sqrt{3}}\,.$$

Or thus,

$$\frac{1}{x\,(1 + x)^2\,(1 + x + x^2)}$$

$$= \frac{1 + x + x^2 - x - x^2}{x\,(1 + x)^2\,(1 + x + x^2)} = \frac{1}{x\,(1 + x)^2} - \frac{1}{(1 + x)\,(1 + x + x^2)}$$

$$= \frac{1 + x - x}{x\,(1 + x)^2} - \left\{ \frac{1 + x + x^2 - x - x^2}{(1 + x)\,(1 + x + x^2)} \right\}$$

$$= \frac{1}{x(1+x)} - \frac{1}{(1+x)^3} - \frac{1}{1+x} + \frac{x}{(1+x+x^2)}$$

$$= \frac{1+x-x}{x(1+x)} - \frac{1}{(1+x)^2} - \frac{1}{(1+x)} + \frac{x}{1+x+x^2}$$

$$= \frac{1}{x} - \frac{2}{1+x} - \frac{1}{(1+x)^2} + \frac{x}{(1+x+x^2)},$$

$$\therefore \int \frac{dx}{x(1+x)^2(1+x+x^2)} =$$

$$\frac{1}{x+1} + \log\left\{\frac{x}{(x+1)^2}\right\} + \int \frac{x\,dx}{(1+x+x^2)}.$$

And $\displaystyle\int \frac{x\,dx}{(1+x+x^2)} = \int \left\{ \frac{(x+\frac{1}{2}) - \frac{1}{2}}{(x+\frac{1}{2})^2 + \frac{3}{4}} \right\} dx$

$$= \log\sqrt{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right),$$

$$\therefore \int \frac{dx}{x(1+x)^2(1+x+x^2)} =$$

$$\frac{1}{x+1} + \log\frac{\sqrt{x^2+x^1+x^1}}{(x+1)^2} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

(8.) $\displaystyle\int \frac{dx}{x^9 + x^7 - x^4 - x^3} = \int \frac{dx}{x^3(x+1)^2(x^2+1)(x-1)}.$

Let $\displaystyle\frac{1}{x^3(x+1)^3(x^2+1)(x-1)} =$

$$\frac{A+Bx}{(x^2+1)} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)} + \frac{E}{(x-1)} + \frac{P}{x^3},$$

$$\therefore 1 = (A + Bv)(x-1)(x+1)^2 x^3 +$$

$$C(x^2+1)(x-1)x^3 + D(x^2+1)(x+1)(x-1)x^3 +$$

$$E(x^2+1)(x+1)^2 x^3 + P(x^2+1)(x+1)^2(x-1).$$

Let $x = \sqrt{-1}$,

$$\therefore 1 = (A+B\sqrt{-1})(1+\sqrt{-1})^2(\sqrt{-1}-1) \times -\sqrt{-1} =$$

$$(A + B\sqrt{-1})(2\sqrt{-1}-2) =$$

$$2A\sqrt{-1} - 2A - 2\sqrt{-1}B - 2B,$$

$\therefore$ by equating $2A\sqrt{-1} = 2B\sqrt{-1}$, $\quad \therefore A = B$,

$$2A + 2B = -1, \qquad \therefore A = B = -\tfrac{1}{4},$$

$$\therefore 1 + \frac{1}{4}(x+1)^3(x-1)x^3 = \frac{1}{4}(x^7 + 2x^6 - 2x^4 - x^3 + 4)$$

$$= \frac{1}{4}(x^2+1)(x^5 + 2x^4 - x^3 - 4x^2 + 4)$$

$$= (x^2+1)\{C(x-1)x^3 + D(x+1)(x-1)x^3$$

$$+ E(x+1)^2 x^3 + P(x+1)^2(x-1)\},$$

$$\therefore \frac{1}{4}(x^5 + 2x^4 - x^3 - 4x^2 + 4) =$$

$$C(x-1)x^3 + D(x+1)(x-1)x^3 + E(x+1)^2 x^3$$

$$+ P(x+1)^2(x-1).$$

Let $x = -1$, $\quad \therefore \frac{1}{2} = 2C$, $\quad \therefore C = \frac{1}{4}$,

$$\therefore \frac{1}{4}(x^5 + 2x^4 - x^3 - 4x^2 + 4 - x^4 + x^3)$$

$$= \frac{1}{4}(x^5 + x^4 - 4x^2 + 4) = \frac{1}{4}(x+1)(x^4 - 4x + 4)$$

$$= (x+1)\{D(x-1)x^3 + E(x+1)x^3 + P(x+1)(x-1)\},$$

$$\therefore \frac{1}{4}(x^4 - 4x + 4)$$

$$= D(x-1)x^3 + E(x+1)x^3 + P(x+1)(x-1).$$

Let $x = -1$, $\therefore \frac{9}{4} = 2D$, $\therefore D = \frac{9}{8}$,

$$\therefore \frac{1}{4}(x^4 - 4x + 4) - \frac{9}{8}(x^4 - x^3) = -\frac{1}{8}(7x^4 - 9x^3 + 8x - 8)$$

$$= -\frac{1}{8}(x+1)(7x^3 - 16x^2 + 16x - 8)$$

$$= (x+1)\{Ex^3 + P(x-1)\},$$

$$\therefore -\frac{1}{8}(7x^3 - 16x^2 + 16x - 8) = Ex^3 + P(x-1).$$

Let $x = 1$, $\therefore E = \frac{1}{8}$,

$$\therefore -\frac{1}{8}(8x^3 - 16x^2 + 16x - 8) = P(x-1),$$

$$\therefore -(x-1)(x^2 - x + 1) = P(x-1),$$

$$\therefore P = -(x^2 - x + 1),$$

$$\therefore \frac{1}{x^5 + x^7 - x^4 - x^3} = -\frac{1}{4}\frac{1+x}{1+x^2}$$

$$+ \frac{1}{4} \cdot \frac{1}{(1+x)^2} + \frac{9}{8} \cdot \frac{1}{1+x} + \frac{1}{8} \cdot \frac{1}{x-1} - \frac{x^2 - x + 1}{x^3},$$

$$\therefore \int \frac{dx}{x^9 + x^7 - x^4 - x^3} = -\frac{1}{4} \int \frac{1+x}{1+x^2} dx$$

$$+ \frac{1}{4} \int \frac{dx}{(1+x)^3} + \frac{9}{8} \int \frac{dx}{1+x}$$

$$+ \frac{1}{8} \int \frac{dx}{x-1} - \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{dx}{x^3}$$

$$= -\frac{1}{8} \int \frac{2x\,dx}{1+x^2} - \frac{1}{4} \int \frac{dx}{x^2+1} + \frac{1}{4} \int \frac{dx}{x+1)^2}$$

$$+ \frac{1}{8} \log (x+1)^9 + \frac{1}{8} \log (x-1) - \log x - \frac{1}{x} + \frac{1}{2\,x^2}$$

$$= \frac{1}{2\,x^2} - \frac{1}{x} - \frac{1}{8} \log (1+x^2) - \frac{1}{4} \tan^{-1} x - \frac{1}{4} \cdot \frac{1}{(x+1)}$$

$$+ \frac{1}{8} \{ \log (x+1)^9 + \log (x-1) - \log x^3 \}$$

$$= \frac{2 + 2x - 4x^2 - 4x - x^2}{4\,x^2\,(1+x)}$$

$$+ \frac{1}{8} \{ \log (x+1)^9 + \log (x-1) - \log (1+x^2) - \log x^8 \}$$

$$- \frac{1}{4} \tan^{-1} x = \frac{2 - 2x - 5x^2}{4\,x^2\,(1+x)}$$

$$+ \log \sqrt[8]{\frac{(x+1)^9\,(x-1)}{(1+x^2)\,x^8}} - \frac{1}{4} \tan^{-1} x.$$

(9.) $\int \frac{(5x-2)\,dx}{x^3 + 6x^2 + 8x}$

$$\frac{5x-2}{x^3 + 6x^2 + 8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}$$

$$5x - 2 = A\,(x+2)\,(x+4) + B\,x\,(x+4) + C\,x\,(x+2).$$

$$\text{Let } x = \quad 0 \quad -2 = \quad 8\,\text{A}, \quad \therefore \text{A} = -\frac{1}{4}$$

$$x = -2 \quad -12 = -4\,\text{B}, \quad \therefore \text{B} = \quad 3$$

$$x = -4 \quad -22 = \quad 8\,\text{C}, \quad \therefore \text{C} = -\frac{11}{4},$$

$$\therefore \int \frac{(5\,x - 2)\,dx}{x^3 + 6x^2 + 8x}$$

$$= -\frac{1}{4}\int \frac{dx}{x} + 3\int \frac{dx}{x + 2} - \frac{11}{4}\int \frac{dx}{x + 4}$$

$$= -\frac{1}{4}\log x + \frac{12}{4}\log(x + 2) - \frac{11}{4}\log(x + 4)$$

$$= \frac{1}{4}\log \frac{(x + 2)^{12}}{x\,(x + 4)^{11}}.$$

(10.) $\int \dfrac{(3\,x + 1)\,dx}{x^3 + 2x^2 + x}$

$$\frac{3\,x + 1}{x^3 + 2\,x^2 + x} = \frac{\text{A}}{x} + \frac{\text{B}}{(x + 1)^2} + \frac{\text{C}}{x + 1},$$

$$3\,x + 1 = \text{A}(x + 1)^2 + \text{B}x + \text{C}x\,(x + 1).$$

Let $x = 0, \quad 1 = \text{A},$

$$3\,x + 1 - x^2 - 2x - 1 = x(1 - x) = \text{B}x + \text{C}x\,(x + 1),$$

$$\therefore 1 - x = \text{B} + \text{C}\,(x + 1).$$

Let $x = -1, \quad 2 = \text{B},$

$$\therefore -(1 + x) = \text{C}(x + 1), \quad \therefore \text{C} = -1,$$

$$\therefore \int \frac{(3\,x + 1)\,dx}{x^3 + 2\,x^2 + x} = \int \frac{dx}{x} + 2\int \frac{dx}{(x + 1)^2} - \int \frac{dx}{x + 1}$$

$$= \log x - \frac{2}{x+1} - \log(x+1)$$

$$= \log \frac{x}{x+1} - \frac{2}{x+1}.$$

(11.) $\int \frac{dx}{1+3x+2x^2}$

Let $\dfrac{1}{1+3x+2x^2} = \dfrac{A}{x+1} + \dfrac{B}{2x+1}$,

$$1 = A(2x+1) + B(x+1).$$

Let $x = -1$ $1 = -A$

$\quad x = -\dfrac{1}{2} \qquad 1 = +\dfrac{B}{2} \qquad B = 2,$

$$\therefore \int \frac{dx}{1+3x+2x^2} = -\int \frac{dx}{x+1} + \int \frac{2\,dx}{2x+1}$$

$$= -\log(x+1) + \log(2x+1) = \log \frac{2x+1}{x+1}.$$

(12.) $\int \dfrac{x\,dx}{(x-2)(x+3)^2}$

Let $\dfrac{x}{(x-2)(x+3)^2} = \dfrac{A}{(x+3)^2} + \dfrac{B}{x+3} + \dfrac{C}{x-2},$

$$x = A(x-2) + (x+3)\{B(x-2) + C(x+3)\}.$$

Let $x = -3$, $\qquad \therefore -3 = -5A$, $\quad A = \dfrac{3}{5}$,

$$\therefore \frac{5x - 3x + 6}{5} = \frac{2(x+3)}{5}$$

$$= (x+3)\{B(x-2) + C(x+3)\}$$

$$\therefore \frac{2}{5} = B(x-2) + C(x+3).$$

Let $x = -3$; $\dfrac{2}{5} = -5B$, $\therefore B = -\dfrac{2}{25}$,

$x = 2$; $\dfrac{2}{5} = 5C$, $\therefore C = \dfrac{2}{25}$,

$$\therefore \int \frac{x\,dx}{(x-2)\,(x+3)^2}$$

$$= \frac{3}{5} \int \frac{dx}{(x+3)^2} - \frac{2}{25} \int \frac{dx}{x+3} + \frac{2}{25} \int \frac{dx}{x-2}$$

$$= -\frac{3}{5} \frac{1}{x+3} - \frac{2}{25} \log (x+3) + \frac{2}{25} \log (x-2)$$

$$= \frac{2}{25} \log \frac{x-2}{x+3} - \frac{3}{5} \frac{1}{x+3}.$$

(13.) $\displaystyle\int \frac{(x^2 + 3x + 1)\,dx}{x^3 + x^2 - 2x}$

Let $\dfrac{x^2 + 3x + 1}{x^3 + x^2 - 2x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$,

$x^2 + 3x + 1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$

Let $x = $ 0 $1 = -2A$, $\therefore A = -\dfrac{1}{2}$

$x = 1$ $5 = 3B$, $\therefore B = \dfrac{5}{3}$

$x = -2$ $-1 = 6C$, $\therefore C = -\dfrac{1}{6}$,

$$\therefore \int \frac{(x^2 + 3x + 1)\,dx}{x^3 + x^2 - 2x}$$

$$= -\frac{1}{2} \int \frac{dx}{x} + \frac{5}{3} \int \frac{dx}{x-1} - \frac{1}{6} \int \frac{dx}{x+2}$$

$$= -\frac{1}{2}\log x + \frac{5}{3}\log (x-1) - \frac{1}{6}\log (x+2)$$

$$= \frac{1}{3}\log \frac{(x-1)^5}{\sqrt{x^4 + 2x^3}}$$

(14.) $\displaystyle\int \frac{x\,dx}{(x+1)\,(x+2)\,(x^2+1)}$

Let $\dfrac{x}{(x+1)(x+2)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{P}{x^2+1}$

$x = A(x+2)(x^2+1) + B(x+1)(x^2+1) + P(x+1)(x+2).$

Let $x = -1$; $\quad -1 = \quad 2A, \quad \therefore A = -\dfrac{1}{2}$

$\quad x = -2$; $\quad -2 = -5B, \quad \therefore B = +\dfrac{2}{5}$

$$x + \frac{x^3 + 2x^2 + x + 2}{2} - \frac{2x^3 + 2x^2 + 2x + 2}{5}$$

$$= \frac{x^3 + 6x^2 + 11x + 6}{10} = P(x+1)(x+2),$$

$$\therefore P = \frac{x+3}{10},$$

$$\therefore \int \frac{x\,dx}{(x+1)\,(x+2)\,(x^2+1)}$$

$$= -\frac{1}{2}\int \frac{dx}{x+1} + \frac{2}{5}\int \frac{dx}{x+2} + \frac{1}{10}\int \frac{x+3}{x^2+1}\,dx$$

$$= -\frac{1}{2}\log (x+1) + \frac{2}{5}\log (x+2) + \frac{1}{20}\log (x^2+1)$$

$$+ \frac{3}{10}\tan^{-1} x,$$

$$= \log \frac{(x^2 + 1)^{\frac{1}{20}} (x + 2)^{\frac{2}{5}}}{\sqrt{x + 1}} + \frac{3}{10} \tan^{-1} x.$$

(15.) $\displaystyle\int \frac{dx}{x^4 + 4x + 3}$

Let $\dfrac{1}{x^4 + 4x + 3} = \dfrac{A}{(x + 1)^2} + \dfrac{B}{x + 1} + \dfrac{Mx + N}{x^2 - 2x + 3}$

$$1 = A (x^2 - 2x + 3) + B(x + 1) (x^2 - 2x + 3)$$
$$+ (Mx + N) (x + 1)^2.$$

Let $x = -1$; $1 = 6A$ $\therefore A = \dfrac{1}{6}$

$$\frac{6 - x^2 + 2x - 3}{6} = - \frac{(x^2 - 2x - 3)}{6} = - \frac{(x + 1)(x - 3)}{6}$$

$$= B(x + 1) (x^2 - 2x + 3) + (Mx + N) (x + 1)^2,$$

$$\therefore - \frac{x - 3}{6} = B (x^2 - 2x + 3) + (Mx + N) (x + 1).$$

Let $x = -1$; $\dfrac{2}{3} = 6B$ $\therefore B = \dfrac{1}{9}$,

$$\therefore \frac{-9x + 27 - 6x^2 + 12x - 18}{54}$$

$$= \frac{-(2x^2 - x - 3)}{18} = (Mx + N) (x + 1)$$

$$Mx + N = \frac{3 - 2x}{18},$$

$$\therefore \int \frac{dx}{x^4 + 4x + 3} =$$

$$\frac{1}{6} \int \frac{dx}{(x+1)} = + \frac{1}{9} \int \frac{dx}{x+1} + \frac{1}{6} \int \frac{dx}{x^2 - 2x + 3}$$

$$- \frac{1}{9} \int \frac{x\,dx}{x^2 - 2x + 3}$$

$$= - \frac{1}{6} \frac{1}{x+1} + \frac{1}{9} \log(x+1) - \frac{1}{18} \log(x^2 - 2x + 3)$$

$$+ \left(\frac{1}{6} - \frac{1}{9} \right) \int \frac{dx}{x^2 - 2x + 3}$$

$$= - \frac{1}{6} \frac{1}{x+1} + \frac{1}{9} \log \frac{x+1}{\sqrt{x^2 - 2x + 3}}$$

$$+ \frac{1}{18\sqrt{2}} \tan^{-1} \frac{x-1}{\sqrt{2}}.$$

(16.) $\int \dfrac{dx}{x(1+x^3)} = \left(\text{putting} \dfrac{1}{z} = x \right) - \int \dfrac{z^4}{z^3 + 1} \cdot \dfrac{dz}{z^2}$

$$= - \frac{1}{3} \int \frac{3z^2\,dz}{z^3 + 1} = - \frac{1}{3} \log(z^3 + 1) = \frac{1}{3} \log \frac{x^3}{x^3 + 1}$$

$$= \log \sqrt[3]{\frac{x^3}{x^3 + 1}}.$$

(17.) $\int \dfrac{dx}{x(1+x^4)} = \left(\text{putting} \dfrac{1}{z} = x \right) - \int \dfrac{z^5\,dz}{z^4 + 1} \cdot \dfrac{1}{z^2}$

$$= - \int \frac{z^3\,dz}{z^4 + 1} = - \frac{1}{4} \log(z^4 + 1) = \log \sqrt[4]{\frac{x^4}{x^4 + 1}}.$$

(18.) $\int \dfrac{dx}{x^1(a+bx^3)} = \left(\text{ if } \dfrac{1}{z} = x \right) - \int \dfrac{z^7}{az^3+b} \cdot \dfrac{dz}{z^2}$

$$= -\int \dfrac{z^5 dz}{az^3+b} = -\int \dfrac{z^2 dz}{a} - \dfrac{b}{a^2}\int \dfrac{az^2 dz}{az^3+b}$$

$$= -\dfrac{z^3}{3a} + \dfrac{b}{3a^2} \log (az^3+b)$$

$$= -\dfrac{1}{3ax^1} + \dfrac{b}{3a^2} \log \left(\dfrac{a+bx^3}{x^1} \right).$$

(19.) $\int \dfrac{dx}{x(1+x^1)^2} \cdot$ Let $x = \dfrac{1}{z}, \;\; dx = -\dfrac{1}{z^2} dz$

$$\therefore \int \dfrac{dx}{x(1+x)^2} = -\int \dfrac{z^7}{(z^3+1)^2} \dfrac{dz}{z^2} = -\int \dfrac{z^5 dz}{(z^3+1)^2}$$

$$= -\int \dfrac{z^2 dz}{(z^3+1)} + \int \dfrac{z^2 dz}{(z^3+1)^2}$$

$$= -\dfrac{1}{3} \log (z^3+1) - \dfrac{1}{3(z^3+1)}$$

$$= -\sqrt[3]{\dfrac{x^3+1}{x^3}} - \dfrac{x^1}{3(x^3+1)}.$$

The following method of doing the last four integrals is very simple, and can be often used with advantage.

$$\int \dfrac{dx}{x(x^1+1)}$$

$$\dfrac{1}{x(x^1+1)} = \dfrac{(1+x^1)-x^1}{x(x^1+1)}$$

$$= \dfrac{1}{x} - \dfrac{x^2}{x^3+1},$$

$$\therefore \int \dfrac{dx}{x(x^1+1)} = \log x - \dfrac{1}{3} \log (x^3+1)$$

$$= \log \sqrt[3]{\dfrac{x^1}{x^1+1}}.$$

And $\int \dfrac{dx}{x(1+x^4)}$ may be found in a similar manner

$$\frac{1}{x(1+x^4)} = \frac{(1+x^4)-x^4}{x(1+x^4)} = \frac{1}{x} - \frac{x^3}{(1+x^4)},$$

$$\therefore \int \frac{dx}{x(1+x^4)} = \log \sqrt[4]{\frac{x^4}{x^4+1}}$$

$$\int \frac{dx}{x^4(a+bx^3)}.$$

Here $\dfrac{1}{x^4(a+bx^3)} = \dfrac{1}{a}\left(\dfrac{a+bx^3-bx^3}{x^4(a+bx^3)}\right)$

$$= \frac{1}{ax^4} - \frac{b}{a}\cdot\frac{1}{x(a+bx^3)}$$

$$\frac{1}{x(a+bx^3)} = \frac{1}{a}\cdot\frac{a+bx^3-bx^3}{x(a+bx^3)}$$

$$= \frac{1}{ax} - \frac{bx^2}{a(a+bx^3)}.$$

Therefore, the original quantity is reduced to

$$\frac{1}{ax^4} - \frac{b}{a^2x} + \frac{b^2}{a^2}\cdot\frac{x^2}{(a+bx^3)},$$

and its integral is $- \dfrac{1}{3ax^3} + \dfrac{b}{3a^2}\log\left(\dfrac{a+bx}{x^3}\right)$

$$\int \frac{dx}{x(1+x^3)^2}.$$

$$\frac{1}{x(1+x^3)^2} = \frac{(1+x^3)-x^3}{x(1+x^3)^2}$$

$$= \frac{1}{x(1+x^3)} - \frac{x^3}{(1+x^3)^2}$$

$$\frac{1}{x(1+x^3)} = \frac{1+x^3-x^3}{x(1+x^3)} = \frac{1}{x} - \frac{x^2}{x^3+1}.$$

Therefore $\dfrac{1}{x\,(1+x^3)^2}$

$$= \frac{1}{x} - \frac{x^2}{x^3+1} - \frac{x^2}{(x^3+1)^2};$$

and the integral required is therefore

$$\log\sqrt[3]{\frac{x^3}{x^3+1}} + \frac{1}{3\,(x^2+1)}.$$

(20.) $\displaystyle\int \frac{dx}{x^6-1}.$

$$\int \frac{dx}{x^6-1} = \frac{1}{2}\int\left\{\frac{dx}{x^3-1} - \frac{dx}{x^3+1}\right\},$$

$$\int\frac{dx}{x^3-1} = \int\frac{dx}{(x-1)\,(x^2+x+1)}.$$

Let $\dfrac{1}{x^3-1} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1},$

$$1 = A\,(x^2+x+1) + Bx + C\,(x-1).$$

Let $x^2+x+1 = 0,$ or $x = \dfrac{\sqrt{-3}-1}{2}.$

$$1 = \left(\frac{B\sqrt{-3}-B}{2} + C\right)\left(\frac{\sqrt{-3}-3}{2}\right),$$

$$\therefore\ 4 = (2C - 4B)\sqrt{-3} - 6C,$$

$$\therefore\ C = -\frac{2}{3}, \quad B = -\frac{1}{3}.$$

Let $x = 1,$ $\therefore\ 1 = 3A,$ $A = \dfrac{1}{3},$

∴ the integral is reduced to

$$\frac{1}{3}\int \left\{ \frac{dx}{x-1} - \frac{(x+2)\,dx}{x^2+x+1} \right\}$$

$$\int \frac{dx}{x-1} = \log(x-1).$$

$$\int \frac{(x+2)\,dx}{x^2+x+1} = \left(\text{if } x = z - \frac{1}{2} \right) \int \frac{\left(z - \frac{1}{2} + 2\right)dz}{\left(z^2 + \frac{3}{4}\right)}$$

$$= \int \frac{\left(z + \frac{3}{2}\right)dz}{z^2 + \frac{3}{4}}$$

$$= \log \sqrt{z^2 + \frac{3}{4}} + \frac{3}{\sqrt{3}} \tan^{-1}\frac{2z}{\sqrt{3}}$$

$$= \log \sqrt{x^2+x+1} + \frac{3}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

$$\therefore \int \frac{dx}{x^3-1} = \frac{1}{3}\log(x-1) - \frac{1}{3}\log\sqrt{x^2+x+1}$$

$$- \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

And in a similar manner it may be shown that

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\log(x+1) - \frac{1}{3}\log\sqrt{x^2-x+1}$$

$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right),$$

$$\therefore \int \frac{dx}{x^6-1} = \frac{1}{6}\log\left(\frac{x-1}{x+1}\frac{\sqrt{x^2-x+1}}{\sqrt{x^2+x+1}}\right)$$

$$- \frac{1}{2\sqrt{3}}\left\{ \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right\}.$$

But these inverse tangents together are equal to

$$\tan^{-1}\left(\frac{\dfrac{4x}{\sqrt{3}}}{1-\left(\dfrac{4x^2-1}{3}\right)}\right) = \tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right)$$

$$\therefore \int\frac{dx}{x^6-1} = \frac{1}{6}\log\left(\frac{x-1}{x+1}\frac{\sqrt{x^2-x+1}}{\sqrt{x^2+x+1}}\right)$$

$$-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right),$$

(21.) $\displaystyle\int\frac{x^5\,dx}{(x^2+1)^3}$,

$$\frac{x^5}{(x^2+1)^3} = \frac{(1+x^2)\,x^3 - x^3}{(x^2+1)^3}$$

$$= \frac{x^3}{(x^2+1)^2} - \frac{x^3}{(x^2+1)^3}.$$

And　$\dfrac{x^3}{(x^2+1)^2} = \dfrac{x(x^2+1)-x}{(x^2+1)^2}$

$$= \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

Also　$\dfrac{x^3}{(1+x^2)^3} = \dfrac{x(x^2+1)-x}{(x^2+1)^3}$

$$= \frac{x}{(x^2+1)^2} - \frac{x}{(x^2+1)^3},$$

$$\therefore \frac{x^5}{(x^2+1)^3} = \frac{x^3}{(x^2+1)^2} - \frac{x^3}{(x^2+1)^3}$$

$$= \frac{x}{x^2+1} - \frac{2x}{(x^2+1)^2} + \frac{x}{(x^2+1)^3};$$

and the integral required is therefore

$$\log \sqrt{x^2 + 1} \ + \ \frac{1}{x^2 + 1} \ - \ \frac{1}{4\,(x^2 + 1)^2},$$

$$= \frac{4\,x^2 + 3}{4\,(x^2 + 1)^2} + \log \sqrt{x^2 + 1}.$$

(22.) $\displaystyle\int \frac{x^2\,dx}{x^4 + 1}$,

$$(x^m + 1) = \left(x^2 - 2x\,\cos\frac{\pi}{m} + 1 \right) \left(x^2 - 2x\,\cos\frac{3\,\pi}{m} + 1 \right)\cdots$$

continued to the factor $\left(x^2 - 2x\,\cos\dfrac{m-1}{m}\,\pi + 1 \right)$ when m is an even number.

This gives

$$(x^4 + 1) = \left(x^2 - 2x\,\cos\frac{\pi}{4} + 1 \right) \left(x^2 - 2x\,\cos\frac{3\,\pi}{4} + 1 \right),$$

or $(x^4 + 1) = (x^2 - x\sqrt{2} + 1)\,(x^2 + x\,\sqrt{2} + 1)$.

Since, $\dfrac{\pi}{4} = 45^\circ \cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}, \quad \cos\dfrac{3\,\pi}{4} = -\sin\dfrac{\pi}{4}$.

Assume therefore

$$\frac{x^2}{x^4 + 1} = \frac{A\,x + B}{x^2 - x\sqrt{2} + 1} \ + \ \frac{C\,x + D}{x^2 + x\sqrt{2} + 1};$$

$$\therefore x^2 = (A\,x + B)\,(x^2 + x\sqrt{2} + 1) + (C\,x + D)\,(x^2 - x\sqrt{2} + 1).$$

If $x^2 + x\sqrt{2} + 1 = 0, \quad x = \dfrac{\sqrt{-1} - 1}{\sqrt{2}}$ (1.)

If $x^2 - x\sqrt{2} + 1 = 0, \quad x = \dfrac{\sqrt{-1} + 1}{\sqrt{2}}$ (2.)

Making (1) our supposition, we have

$$x^2 = -\frac{2\sqrt{-1}}{2} = -\sqrt{-1} = \frac{C\sqrt{-1}-C}{\sqrt{2}} + D\cdot(2-2\sqrt{-1}),$$

or, $-\sqrt{-1} = \dfrac{2C\sqrt{-1}}{\sqrt{2}} + \dfrac{2C\sqrt{-1}}{\sqrt{2}} - \dfrac{2C}{\sqrt{2}} + \dfrac{2C}{\sqrt{2}}$

$$+ 2D - 2D\sqrt{-1}\cdot\sqrt{-1} = \left\{\frac{4C}{\sqrt{2}} - 2D\right\}\sqrt{-1} + 2D,$$

$$\therefore D = 0 \quad C = -\frac{1}{2\sqrt{2}}.$$

Now making (2) our supposition, we have

$$x^2 = \sqrt{-1} = \left\{\left(\frac{A\sqrt{-1}+A}{\sqrt{2}}\right) + B\right\}\left(2 + 2\sqrt{-1}\right),$$

$=$ (by reduction, as in the preceding case,)

$$\left\{\frac{4A}{\sqrt{2}} + 2B\right\}\sqrt{-1} + 2B, \quad \therefore B = 0, \ A = \frac{1}{2\sqrt{2}}.$$

The quantity under consideration is now reduced to

$$\frac{1}{2\sqrt{2}}\left\{\frac{x}{x^2 - x\sqrt{2}+1} - \frac{x}{x^2 + x\sqrt{2}+1}\right\}.$$

The integrals of $\dfrac{x}{x^2 - x\sqrt{2}+1}$ and $\dfrac{x}{x^2 + x\sqrt{2}+1}$ now re-

main to be found. They can both be included in the general

case $\dfrac{x}{x^2 \pm x\sqrt{2}+1}$,

$$\int\frac{x\,dx}{x^2 \pm x\sqrt{2}+1} = \left(\text{if } x = z \mp \frac{1}{\sqrt{2}}\right) \int\frac{\left(z \mp \frac{1}{\sqrt{2}}\right)dz}{z^2 + \frac{1}{2}}$$

$$= \log\sqrt{z^2 + \frac{1}{2}}$$

$$\mp \tan^{-1}(z\sqrt{2}). \quad \text{This gives} \int \frac{x\,dx}{x^2 - x\sqrt{2} + 1}$$

$$= \log\sqrt{(x^2 - x\sqrt{2} + 1)} + \tan^{-1}(x\sqrt{2} - 1),$$

$$\int \frac{x\,dx}{x^2 + x\sqrt{2} + 1} = \log\sqrt{(x^2 + x\sqrt{2} + 1)}$$

$$- \tan^{-1}(x\sqrt{2} + 1), \quad \therefore 2\sqrt{2}\int_x \frac{x^2}{x^4 + 1} =$$

$$\log\sqrt{\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}} + \tan^{-1}(x\sqrt{2} - 1)$$

$$+ \tan^{-1}(x\sqrt{2} + 1).$$

And these inverse tangents may be reduced thus:

$$\tan^{-1}(x\sqrt{2} - 1) + \tan^{-1}(x\sqrt{2} + 1)$$

$$= \tan^{-1}\left(\frac{x\sqrt{2} - 1 + x\sqrt{2} + 1}{1 - (x\sqrt{2} - 1)(x\sqrt{2} + 1)}\right)$$

$$= \tan^{-1}\frac{2x\sqrt{2}}{1 - (2x^2 - 1)} = \tan^{-1}\frac{2x\sqrt{2}}{2(1 - x^2)} = \tan^{-1}\frac{x\sqrt{2}}{(1 - x^2)}.$$

$$\therefore \int \frac{x^2\,dx}{x^4 + 1} =$$

$$\frac{1}{2\sqrt{2}}\log\sqrt{\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}} + \frac{1}{2\sqrt{2}}\tan^{-1}\left\{\frac{x\sqrt{2}}{1 - x^2}\right\}.$$

(23.) $\displaystyle\int \frac{x^6\,dx}{x^3 + 1}$

$$\frac{x^6}{x^3 + 1} = x^3 - 1 + \frac{1}{x^3 + 1}$$

$$\frac{1}{x^3 + 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1},$$

$$\therefore 1 = A\left(x^2 - x + 1\right) + Bx + C\left(x + 1\right)$$

$$x = -1, \qquad \therefore 1 = 3A, \qquad \therefore A = \frac{1}{3},$$

$$\therefore \frac{3 - (x^2 - x + 1)}{x + 1} = -\frac{x^2 + x + 2}{x + 1}$$

$$= -\frac{x^2 - x + 2x + 2}{(x + 1)} = -x + 2 = Bx + C,$$

$$\therefore \frac{1}{x^3 + 1} = \frac{1}{3(x + 1)} + \frac{2 - x}{x^2 - x + 1}$$

$$\int \frac{(2 - x)\,dx}{x^2 - x + 1} = \left(\text{if } x = z + \frac{1}{2}\right) \int \frac{\left\{2 - \left(z + \frac{1}{2}\right)\right\}dz}{z^2 + \frac{3}{4}}$$

$$= \int \frac{\frac{3}{2}\,dz}{z^2 + \frac{3}{4}} - \int \frac{z\,dz}{z^2 + \frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2z}{\sqrt{3}}\right) - \log \sqrt{z^2 + \frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) - \log \sqrt{x^2 - x + 1},$$

$$\text{and } \int \frac{dx}{3(x + 1)} = \frac{1}{3} \log (x + 1),$$

$$\therefore \int \frac{dx}{x^3 + 1} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) +$$

$$\log \frac{\sqrt[3]{x + 1}}{\sqrt{x^2 - x + 1}}.$$

Also, $\int x^3 dx = \dfrac{x^4}{4}$, $\int dx = x$,

therefore the original integral is

$$\frac{x^4}{4} - x + \frac{1}{\sqrt{3}} \tan^{-1} \left\{ \frac{2x-1}{\sqrt{3}} \right\} + \log \frac{\sqrt[3]{x+1}}{\sqrt{x^2 - x + 1}}$$

The method of integrating by parts is used to great advantage in many integrals, which is as follows :—

$$d(pq) = p\,dq + q\,dp,$$

$$\therefore p\,dq = d(pq) - q\,dp$$

$$\int p\,dq = pq - \int q\,dp \ldots (1);$$

or by using differential coefficients

$$\int p\,\frac{dq}{dx} = pq - \int q\,\frac{dp}{dx} \ldots (2).$$

In the following examples we shall sometimes use (1) and sometimes (2).

When integrals are of the form of $\int x^{m-1}(a + bx^n)^{\frac{p}{q}}\,dx$,

they can be rationalized by assuming $a + bx^n = z^q$ when

$\dfrac{m}{n}$ or $\dfrac{m}{n} + \dfrac{p}{q}$ is an integer. If $\dfrac{m}{n}$ be a fraction assume $a + bx^n$ $= x^n z^q$.

CHAPTER III.

(1.) $\int x\, dx \sqrt{a+x} = \int dx\, (a+x-a)\, \sqrt{a+x}$

$$= \int dx\, (a+x)^{\frac{3}{2}} - a \int dx \sqrt{a+x}$$

$$= \frac{2}{5} (a+x)^{\frac{5}{2}} - \frac{2a}{3} (a+x)^{\frac{3}{2}}.$$

(2.) $\dfrac{du}{dx} = \dfrac{1}{\sqrt{x+a} + \sqrt{x}} = \dfrac{\sqrt{x+a} - \sqrt{x}}{a},$

$$\therefore\; u = \frac{2}{3a} (x+a)^{\frac{3}{2}} - \frac{2}{3a} x^{\frac{3}{2}}.$$

(3.) $\dfrac{du}{dx} = (a-x)(b-x)^{\frac{m}{n}} = (b-x+a-b)\,(b-x)^{\frac{m}{n}}$

$$= (b-x)^{\frac{m}{n}+1} + (a-b)(b-x)^{\frac{m}{n}},$$

$$\therefore u = -\frac{n(b-x)^{\frac{m}{n}+2}}{m+2n} - \frac{n(a-b)(b-x)^{\frac{m}{n}+1}}{m+n)}.$$

(4.) $\dfrac{du}{dx} = (x^2 + a) \sqrt{x^2 + 4a}$

$\qquad = (x^2 + 4a - 3a) \sqrt{x^2 + 4a}$

$\qquad = (x^2 + 4a)^{\frac{3}{2}} - 3a \sqrt{x^2 + 4a}.$

To integrate $(x^2 + 4a)^{\frac{3}{2}}$ (integrating by parts).

Let $p = (x^2 + 4a)^{\frac{3}{2}}$ $\qquad \dfrac{dq}{dx} = 1$

$\qquad \dfrac{dp}{dx} = 3x(x^2 + 4b)^{\frac{1}{2}}$ $\qquad q = x,$

$\therefore \displaystyle\int dx (x^2 + 4a)^{\frac{3}{2}} = x(x^2 + 4a)^{\frac{3}{2}} - 3 \int x^2 dx (x^2 + 4a)^{\frac{1}{2}}$

$= x(x^2 + 4a)^{\frac{3}{2}} - 3 \displaystyle\int dx(x^2 + 4a)^{\frac{3}{2}} + 12a \int dx(x^2 + 4a)^{\frac{1}{2}},$

$\therefore \displaystyle\int dx(x^2 + 4a)^{\frac{3}{2}} = \dfrac{x}{4}(x^2 + 4a)^{\frac{3}{2}} + 3a \int dx(x^2 + 4a)^{\frac{1}{2}},$

$\therefore u = \dfrac{x}{4}(x^2 + 4a)^{\frac{3}{2}}.$

(5.) $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{x+a}}.$ Let $a + x = z^2$

$\qquad \dfrac{du}{dz} = \dfrac{du}{dx}\,\dfrac{dx}{dz} = \dfrac{1}{(z^2 - a)z} \cdot 2z = \dfrac{2}{z^2 - a}$

$\qquad = \dfrac{1}{\sqrt{a}}\left\{\dfrac{1}{z - \sqrt{a}} - \dfrac{1}{z + \sqrt{a}}\right\}$

$\qquad u = \dfrac{1}{\sqrt{a}}\log\dfrac{z - \sqrt{a}}{z + \sqrt{a}} = \dfrac{1}{\sqrt{a}}\log\dfrac{z^2 - a}{(z + \sqrt{a})^2}$

$$= \frac{2}{\sqrt{a}} \log \frac{\sqrt{x}}{\sqrt{a+x}+\sqrt{a}}.$$

(6.) $\dfrac{d\,u}{d\,x} = \dfrac{\sqrt{a+bx^3}}{x}$.   Let $a + bx^3 = z^2$,

$$x = \left(\frac{z^2-a}{b}\right)^{\frac{1}{3}}, \quad \frac{d\,x}{d\,z} = \frac{1}{b^{\frac{1}{3}}} \frac{2z}{3(z^2-a)^{\frac{2}{3}}}$$

$$\frac{d\,u}{d\,z} = \frac{d\,u}{d\,x} \cdot \frac{d\,x}{d\,z} = \frac{b^{\frac{1}{3}} z}{(z^2-a)^{\frac{1}{3}}} \cdot \frac{2z}{3b^{\frac{1}{3}} (z^2-a)^{\frac{2}{3}}}$$

$$= \frac{2z^2}{3(z^2-a)} = \frac{2}{3}\left\{1 + \frac{a}{z^2-a}\right\}$$

$$= \frac{2}{3}\left\{1 + \frac{\sqrt{a}}{2}\left(\frac{1}{z-\sqrt{a}} - \frac{1}{z+\sqrt{a}}\right)\right\},$$

$$\therefore u = \frac{2z}{3} - \frac{2}{3} \cdot \frac{\sqrt{a}}{2} \log \frac{z+\sqrt{a}}{z-\sqrt{a}}$$

$$= \frac{2z}{3} - \frac{2\sqrt{a}}{3} \log \frac{z+\sqrt{a}}{\sqrt{z^2-a}}$$

$$= \frac{2\sqrt{a+bx^3}}{3} - \frac{2\sqrt{a}}{3} \log \frac{\sqrt{a+bx^3}+\sqrt{a}}{\sqrt{bx^3}}$$

$$= \frac{2\sqrt{a+bx^3}}{3} - \frac{2\sqrt{a}}{3} \log \frac{\sqrt{b+ax^{-3}}+\sqrt{ax^{-3}}}{\sqrt{b}}.$$

(7.) $\dfrac{du}{dx} = \dfrac{1 + x^2}{(1 - x^2)\sqrt{1 + x^4}} = \dfrac{\dfrac{1}{x^2} + 1}{\left(\dfrac{1}{x} - x\right)\sqrt{\dfrac{1}{x^2} + x^2}}$

$$= \dfrac{\dfrac{1}{x^2} + 1}{\left(\dfrac{1}{x} - x\right)\sqrt{\left(\dfrac{1}{x} - x\right)^2 + 2}},$$

$$= \dfrac{-dz}{z\sqrt{z^2 + 2}}\left(\text{ if } \dfrac{1}{x} - x = z\right),$$

$$\therefore u = -\dfrac{1}{\sqrt{2}}\log\left\{\dfrac{z}{\sqrt{z^2 + 2} + \sqrt{2}}\right\}$$

$$= \dfrac{1}{\sqrt{2}}\log\left\{\dfrac{\sqrt{z^2 + 2} + \sqrt{2}}{z}\right\}$$

$$= \dfrac{1}{\sqrt{2}}\log\dfrac{\sqrt{2x} + \sqrt{1 + x^4}}{1 - x^2}.$$

(8.) $\displaystyle\int\dfrac{dx}{\sqrt{x^2 + x + 1}} = \int\dfrac{dx}{\sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}}$

$$= \log\left\{\sqrt{(x + \tfrac{1}{2})^2 + \tfrac{3}{4}} + x + \tfrac{1}{2}\right\}$$

$$= \log\left\{2\sqrt{x^2 + x + 1} + 2x + 1\right\} + C.$$

(9.) $\displaystyle\int\dfrac{dx}{\sqrt{1 + x - x^2}} = \int\dfrac{dx}{\sqrt{\frac{5}{4} - (x - \frac{1}{2})^2}}$

$$= \sin^{-1} \frac{x - \dfrac{1}{2}}{\dfrac{\sqrt{5}}{2}} = \sin^{-1} \frac{2x - 1}{\sqrt{5}}.$$

(10.) $\displaystyle\int \frac{dx}{(1 + x)\sqrt{1 - x}}$ Let $1 - x = z^2$,

$$x = 1 - z^2 \qquad dx = -2z\,dz$$

$$1 + x = 2 - z^2$$

$$\int \frac{dx}{(1 + x)\sqrt{1 - x}} = -2 \int \frac{dz}{2 - z^2}$$

$$= -\frac{2}{2\sqrt{2}} \left\{ \int \frac{dz}{\sqrt{2} + z} + \int \frac{dz}{\sqrt{2} - z} \right\}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2} - z}{\sqrt{2} + z} = \frac{1}{\sqrt{2}} \log \frac{2 + z^2 - 2\sqrt{2}z}{2 - z^2}$$

$$= \frac{1}{\sqrt{2}} \log \frac{3 - x - 2\sqrt{2}\sqrt{1 - x}}{1 + x}.$$

(11.) $\displaystyle\int \frac{dx}{x\sqrt{1 + x + x^2}}$.   Let $x = \dfrac{1}{z}$ $dx = -\dfrac{dz}{z^2}$

$$= -\int \frac{dz}{\sqrt{1 + z + z^2}} = -\int \frac{dz}{\sqrt{(z + \frac{1}{2})^2 + \frac{3}{4}}}$$

$$= -\log \left\{ \sqrt{1 + z + z^2} + z + \tfrac{1}{2} \right\}$$

$$= -\log \left\{ \frac{2\sqrt{1 + x + x^2} + 2 + x}{x} \right\}$$

$$= \log \frac{x}{2\sqrt{1+x+x^2}+2+x}$$

$$= \log \frac{x\,(2+x-2\sqrt{1+x+x^2})}{4+4x+x^2-4-4x-4x^2}$$

$$= \log \frac{2+x-2\sqrt{1+x+x^2}}{-3x}$$

(12.) $\dfrac{d\,u}{d\,x} = \dfrac{x}{\sqrt{a^4-x^4}} = \dfrac{1}{2}\dfrac{\dfrac{d\,(x^2)}{d\,x}}{\sqrt{a^4-(x^2)^2}},$

$$\therefore\; u = \frac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$$

(13.) $\dfrac{d\,u}{d\,x} = \dfrac{1}{\sqrt{(1-x)\,(x+2)}} = \dfrac{1}{\sqrt{2-x-x^2}}$

$$= \frac{2}{\sqrt{9-(2x+1)^2}},$$

$$\therefore\; u = \sin^{-1}\frac{2x+1}{3}.$$

(14.) $\dfrac{d\,u}{d\,x} = \dfrac{x}{\sqrt{(x^2-a^2)\,(b^2-x^2)}}$

$$= \frac{x}{\sqrt{x^2-a^2}}\,\frac{1}{\sqrt{b^2-a^2-(x^2-a^2)}},$$

$$\therefore\; u = \sin^{-1}\sqrt{\frac{x^2-a^2}{b^2-a^2}}.$$

(15.) $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2-a^2}\sqrt{b^2-x^2}}$

Let $x = \dfrac{1}{z}$ $\dfrac{dx}{dz} = -\dfrac{1}{z^2}$,

$$\frac{du}{dz} = \frac{du}{dx} \cdot \frac{dx}{dz} = \frac{z^3}{\sqrt{1 - a^2 z^3}\,\sqrt{b^2 z^2 - 1}} \cdot -\frac{1}{z^2}$$

$$= -\frac{a^2 z}{\sqrt{1 - a^2 z^3}} \; \frac{1}{a^2 \cdot \dfrac{b}{a}\sqrt{1 - \dfrac{a^3}{b^3} - (1 - a^2 z^2)}},$$

$$\therefore u = \frac{1}{a\,b}\,\sin^{-1}\left\{\frac{1 - a^2 z^2}{1 - \dfrac{a^2}{b^2}}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{a\,b}\,\sin^{-1}\frac{b}{x}\sqrt{\frac{x^2 - a^2}{b^2 - a^2}}.$$

(16.) $\quad \dfrac{du}{dx} = \dfrac{1}{(1 + x)\sqrt{1 - x^2}}$

$$= \frac{1}{(1 + x)\sqrt{2(1 + x) - (1 + x)^2}}$$

$$= \frac{(1 + x)^{-2}}{\sqrt{2(1 + x)^{-1} - 1}}.$$

$$\therefore u = -\sqrt{2(1 + x)^{-1} - 1}$$

$$= -\sqrt{\frac{2}{1 + x} - 1} = -\sqrt{\frac{1 - x}{1 + x}}.$$

(17.) $\quad \dfrac{du}{dx} = \dfrac{x}{(1 + x^2)\sqrt{1 - x^4}}$

$$= \frac{x}{(1 + x^2)\sqrt{2(1 + x^2) - (1 + x^2)^2}} = \frac{x(1 + x^2)^{-2}}{\sqrt{2(1 + x^2)^{-1} - 1}},$$

$$\therefore u = -\frac{1}{2}\sqrt{2(1 + x^2)^{-1} - 1} = -\frac{1}{2}\sqrt{\frac{1 - x^2}{1 + x^2}}.$$

(18.) $\dfrac{du}{dx} = \dfrac{\left(x + \sqrt{1 + x^2}\right)^{\frac{m}{n}}}{\sqrt{1 + x^2}}$

$$= \left(\dfrac{x}{\sqrt{1 + x^2}} + 1\right)\left(x + \sqrt{1 + x^2}\right)^{\frac{m}{n} - 1},$$

$\therefore\ u = \dfrac{n}{m}\left(x + \sqrt{1 + x^2}\right)^{\frac{m}{n}}.$

(19.) $\dfrac{du}{dx} = \dfrac{1}{(1 + x)\sqrt{1 + x - x^2}}.$

Let $1 + x = z,\qquad x^2 = (z - 1)^2,$

$$\dfrac{du}{dz} = \dfrac{du}{dx} \cdot \dfrac{dx}{dz} = \dfrac{1}{z\sqrt{z - (z-1)^2}} = \dfrac{1}{z\sqrt{3z - z^2 - 1}}.$$

Let $z = \dfrac{1}{v},\ \dfrac{dz}{dv} = -\dfrac{1}{v^2}$

$$\dfrac{du}{dv} = \dfrac{du}{dz} \cdot \dfrac{dz}{dv} = -\dfrac{1}{v^2}\dfrac{v^2}{\sqrt{3v - 1 - v^2}}$$

$$= \dfrac{-1}{\sqrt{\dfrac{5}{4} - \left(\dfrac{3}{2} - v\right)^2}},$$

$\therefore\ u = -\sin^{-1}\dfrac{\dfrac{3}{2} - v}{\dfrac{\sqrt{5}}{2}} = -\sin^{-1}\dfrac{3 - 2v}{\sqrt{5}}$

$$= -\sin^{-1}\dfrac{3z - 2}{\sqrt{5}z} = -\sin^{-1}\dfrac{3x + 1}{\sqrt{5}(x + 1)}.$$

$$= \tan^{-1}\left(\dfrac{3x + 1}{2\sqrt{1 + x - x^2}}\right).$$

(20.) $\dfrac{du}{dx} = \dfrac{1}{(x+b)\sqrt{x+a}}.$

Let $x + a = z^2$ $\dfrac{dx}{dz} = 2z,$

$$\frac{du}{dz} = \frac{du}{dx} \cdot \frac{dx}{dz} = \frac{1}{(z^2 - a + b)z} \cdot 2z = \frac{2}{z^2 + b - a},$$

$$\therefore \ u = \frac{2}{\sqrt{b-a}} \tan^{-1} \frac{z}{\sqrt{b-a}}$$

$$= \frac{2}{\sqrt{b-a}} \tan^{-1} \sqrt{\frac{x+a}{b-a}}.$$

(21.) $\dfrac{du}{dx} = \dfrac{1}{(x+b)(x+a)^{\frac{3}{2}}}$

$$= \frac{1}{a-b} \left\{ \frac{1}{x+b} - \frac{1}{x+a} \right\} \frac{1}{\sqrt{x+a}},$$

$$= \frac{1}{a-b} \cdot \frac{1}{(x+b)\sqrt{x+a}} - \frac{1}{a-b} \cdot \frac{1}{(x+a)^{\frac{3}{2}}},$$

$$\therefore \ u = -\frac{2}{(b-a)^{\frac{3}{2}}} \tan^{-1} \sqrt{\frac{x+a}{b-a}} + \frac{2}{a-b} \frac{1}{\sqrt{x+a}}$$

(22.) $\dfrac{du}{dx} = \dfrac{1}{x} \sqrt{x^2 - a^2} = \dfrac{1}{x} \dfrac{x^2 - a^2}{\sqrt{x^2 - a^2}}$

$$= \frac{x}{\sqrt{x^2 - a^2}} - \frac{a^2}{x\sqrt{x^2 - a^2}},$$

$$\therefore \ u = \sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a}.$$

(23.) $\dfrac{du}{dx} = \dfrac{\sqrt{a^2 \pm x^2}}{x^2} = \dfrac{a^2}{x^2\sqrt{a^2 \pm x^2}} \pm \dfrac{1}{\sqrt{a^2 \pm x^2}}.$

To integrate $\dfrac{1}{x^2\sqrt{a^2 \pm x^2}}.$

Let $x = \dfrac{1}{z}, \qquad \dfrac{dx}{dz} = -\dfrac{1}{z^2},$

$\dfrac{du}{dz} = \dfrac{dx}{dz} \cdot \dfrac{du}{dx} = -\dfrac{1}{z^2}\dfrac{z^3}{\sqrt{a^2 z^2 \pm 1}} = -\dfrac{z}{\sqrt{a^2 z^2 \pm 1}},$

$\therefore \; u = -\dfrac{\sqrt{a^2 z^2 \pm 1}}{a^2},$

$\therefore \displaystyle\int \dfrac{\sqrt{a^2 \pm x^2}}{x^2}\,dx = -\dfrac{\sqrt{a^2 + x^2}}{x} + \log(x + \sqrt{a^2 + x^2}),$

or, $= -\dfrac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\dfrac{x}{a},$ accordingly as we use $a^2 + x^2$ or $a^2 - x^2$.

(24.) $\dfrac{du}{dx} = \dfrac{1}{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}$

$= \dfrac{\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2}}{2x^2}$

$= \dfrac{1}{2}\dfrac{\sqrt{a^2 + x^2}}{x^2} + \dfrac{1}{2}\dfrac{\sqrt{a^2 - x^2}}{x^2}.$

By last example,

$u = -\dfrac{\sqrt{a^2 + x^2} + \sqrt{a^2 + x^2}}{2x}$

$+ \dfrac{1}{2}\log(x + \sqrt{a^2 + x^2}) - \dfrac{1}{2}\sin^{-1}\dfrac{x}{a}.$

(25.) $\dfrac{du}{dx} = \sqrt{\dfrac{a+x}{a-x}} = \dfrac{a}{\sqrt{a^2-x^2}} + \dfrac{x}{\sqrt{a^2-x^2}}$,

$$\therefore \; u = a \sin^{-1}\dfrac{x}{a} - \sqrt{a^2-x^2}.$$

(26.) $\dfrac{du}{dx} = \dfrac{1}{x}\sqrt{\dfrac{x+a}{x-a}} = \dfrac{1}{\sqrt{x^2-a^2}} + \dfrac{a}{x\sqrt{x^2-a^2}}$,

$$\therefore \; u = \log\left(x + \sqrt{x^2-a^2}\right) + \sec^{-1}\dfrac{x}{a}.$$

(27.) $\dfrac{du}{dx} = x\sqrt{\dfrac{a+x}{a-x}} = \dfrac{ax}{\sqrt{a^2-x^2}} + \dfrac{x^2}{\sqrt{a^2-x^2}}.$

To integrate $\dfrac{x^2}{\sqrt{a^2-x^2}}.$

$$p = x \qquad \dfrac{dq}{dx} = \dfrac{x}{\sqrt{a^2-x^2}},$$

$$\dfrac{dp}{dx} = 1 \qquad q = -\sqrt{a^2-x^2}.$$

$$\therefore \int \dfrac{x^2\,dx}{\sqrt{a^2-x^2}} = -x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,dx$$

$$= -x\sqrt{a^2-x^2} + \int \dfrac{a^2\,dx}{\sqrt{a^2-x^2}} - \int \dfrac{x^2\,dx}{\sqrt{a^2-x^2}}$$

$$\therefore \int \dfrac{x^2\,dx}{\sqrt{a^2-x^2}} = -\dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\int \dfrac{dx}{\sqrt{a^2-x^2}}.$$

$$\therefore\ u = a \int \frac{x\,dx}{\sqrt{a^2-x^2}} + \int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \left(a + \frac{x}{2}\right)\sqrt{a^2-x^2}.$$

(28.) $\quad \dfrac{du}{dx} = \dfrac{1}{\sqrt{1-2acx+a^2c^2}\sqrt{1-2ac^{-1}x+a^2c^{-2}}}.$

Let $1 - 2acx + a^2c^2 = z^2 \qquad \dfrac{dx}{dz} = -\dfrac{z}{ac}$

$$-2ac^{-1}x = \frac{z^2 - (1 + a^2c^2)}{c^2}$$

$$1 - 2ac^{-1}x + a^2c^{-2} = \frac{c^2 + a^2 + z^2 - 1 - a^2c^2}{c^2},$$

$$\therefore\ \frac{du}{dz} = \frac{du}{dx}\cdot\frac{dx}{dz} = -\frac{z}{ac}\cdot\frac{c}{z\sqrt{z^2-(1-a^2-c^2+a^2c^2)}}$$

$$= -\frac{1}{a}\frac{1}{\sqrt{z^2-(1-a^2-c^2+a^2c^2)}},$$

$$\therefore\ u = -\frac{1}{a}\log\left\{z + \sqrt{z^2-(1-a^2-c^2+a^2c^2)}\right\}$$

$$= \frac{1}{a}\log\left\{\sqrt{1-2acx+a^2c^2} + \sqrt{a^2-2acx+c^2}\right\}$$

(29.) $\quad \dfrac{du}{dx} = \dfrac{1}{(1-x^2)\sqrt{1+x^2}}.\qquad$ Let $x = \dfrac{1}{z}$

$$\frac{du}{dz} = -\frac{1}{z^2}\frac{z^3}{(z^2-1)\sqrt{1+z^2}} = -\frac{z}{(z^2-1)\sqrt{1+z^2}}.$$

Let $1 + z^2 = v^2, \quad z^2 - 1 = v^2 - 2, \quad \dfrac{dz}{dv} = \dfrac{v}{\sqrt{v^2-1}}$

$$\frac{du}{dv} = \frac{du}{dz} \cdot \frac{dz}{dv} = -\frac{\sqrt{v^2 - 1}}{(v^2 - 2)\,v} \cdot \frac{v}{\sqrt{v^2 - 1}} = -\frac{1}{v^2 - 2}$$

$$\frac{du}{dv} = \frac{1}{2\sqrt{2}} \left\{ \frac{1}{v + \sqrt{2}} - \frac{1}{v - \sqrt{2}} \right\},$$

$$\therefore\; u = \frac{1}{2\sqrt{2}} \log \frac{v + \sqrt{2}}{v - \sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{(v + \sqrt{2})^2}{v^2 - 2}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{1 + z^2} + \sqrt{2}}{\sqrt{z^2 - 1}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{1 + x^2} + \sqrt{2}\,x}{\sqrt{1 - x^2}}$$

(30.) $\quad \dfrac{du}{dx} = \dfrac{1}{(1 + x^2)\sqrt{1 - x^2}}.$ $\qquad$ Let $x = \dfrac{1}{z}$

$$\frac{du}{dz} = -\frac{z}{(1 + z^2)\sqrt{z^2 - 1}}.$$

Let $z^2 - 1 = v^2$, $\;1 + z^2 = v^2 + 2$, $\;\dfrac{dz}{dv} = \dfrac{v}{\sqrt{v^2 + 1}}$,

$$\therefore\; \frac{du}{dv} = \frac{du}{dz} \cdot \frac{dz}{dx} = -\frac{\sqrt{v^2 + 1}}{(v^2 + 2)\,v} \cdot \frac{v}{\sqrt{v^2 + 1}} = -\frac{1}{v^2 + 2}$$

$$u = -\frac{1}{\sqrt{2}} \tan^{-1} \frac{v}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{z^2 - 1}}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{1 - x^2}}{\sqrt{2}\,x}.$$

(31.) $\displaystyle\int \frac{dx}{x\sqrt{x^2 + x - 1}}$.   Let $x = \dfrac{1}{z}$, $dx = -\dfrac{dz}{z^2}$

$$= -\int \frac{dz}{\sqrt{1 + z - z^2}} = -\int \frac{dz}{\sqrt{\dfrac{5}{4} - \left(z - \dfrac{1}{2}\right)^2}}$$

$$= -\sin^{-1} \frac{z - \dfrac{1}{2}}{\dfrac{\sqrt{5}}{2}} = -\sin^{-1} \frac{2z - 1}{\sqrt{5}}$$

$$= -\sin^{-1} \frac{2 - x}{\sqrt{5}\, x} = \sin^{-1} \frac{x - 2}{\sqrt{5}\, x}.$$

(32.) $\displaystyle\int \frac{dx}{(1 + x)\sqrt{1 + x^2}}$.   Let $1 + x = \dfrac{1}{z}$

$$x = \frac{1}{z} - 1, \qquad dx = -\frac{dz}{z^2}$$

$$1 + x^2 = 2 - \frac{2}{z} + \frac{1}{z^2},$$

$$\therefore \int \frac{dx}{(1 + x)\sqrt{1 + x^2}} = \int \frac{-\dfrac{dz}{z^2}}{\dfrac{1}{z}\sqrt{2 - \dfrac{2}{z} + \dfrac{1}{z^2}}}$$

$$= -\int \frac{dz}{\sqrt{2z^2 - 2z + 1}} = -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{\left(z - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}}}$$

$$= -\frac{1}{\sqrt{2}} \log \left\{ \sqrt{z^2 - z + \frac{1}{2}} + z - \frac{1}{2} \right\}$$

$$= -\frac{1}{\sqrt{2}} \log \left\{ \frac{\sqrt{1 + x^2}}{\sqrt{2}\,(1 + x)} + \frac{1 - x}{2\,(1 + x)} \right\}$$

$$= -\frac{1}{\sqrt{2}} \log \left\{ \frac{\sqrt{2}\sqrt{1+x^2}+1-x}{2(1+x)} \right\}$$

$$= \frac{1}{\sqrt{2}} \log \frac{2(1+x)}{1-x+\sqrt{2}\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{2}} \log \frac{2(1+x)\{1-x-\sqrt{2}\sqrt{1+x^2}\}}{1-2x+x^2-2-2x^2}$$

$$= \frac{1}{\sqrt{2}} \log \left\{ \frac{2(1-x-\sqrt{2}\sqrt{1+x^2})}{-(1+x)} \right\}.$$

(33.) $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.   Let $1-x^2 = x^2 z^2$

$$x^2 = \frac{1}{z^2+1} \qquad 1+x^2 = \frac{2+z^2}{1+z^2}$$

$$2\log x = -\log(z^2+1),$$

$$\frac{dx}{x} = -\frac{z\,dz}{z^2+1}$$

$$\int \frac{dx}{(1+x^2)xz} = -\int \frac{dz}{z^2+1} \times \frac{z^2+1}{z^2+2} = -\int \frac{dz}{2+z^2}$$

$$= \frac{1}{\sqrt{2}} \cot^{-1} \frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cot^{-1} \frac{\sqrt{1-x^2}}{\sqrt{2}\,x}.$$

Let $\theta = \cot^{-1} \dfrac{\sqrt{1-x^2}}{\sqrt{2}\,x}$

$$\cot\theta = \frac{\sqrt{1-x^2}}{\sqrt{2}\,x}$$

$$\cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}} = \frac{\dfrac{\sqrt{1-x^2}}{\sqrt{2}\,x}}{\sqrt{1+\dfrac{1-x^2}{2x^2}}}$$

$$\cos\theta = \sqrt{\frac{1-x^2}{1+x^2}}, \qquad \therefore \theta = \cos^{-1}\sqrt{\frac{1-x^2}{1+x^2}},$$

$$\therefore \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{\sqrt2}\cos^{-1}\sqrt{\frac{1-x^2}{1+x^2}}.$$

(34.) $\displaystyle\int \frac{dx}{(1-x^2)\sqrt{(1+x^2)}}.$ Let $1+x^2 = x^2 z^2$

$$x^2 = \frac{1}{z^2-1} \qquad 1-x^2 = \frac{z^2-2}{z^2-1}$$

$$2\log x = -\log(z^2-1)$$

$$\frac{dx}{xz} = -\frac{dz}{z^2-1}$$

$$\int \frac{dx}{(1-x^2)\,xz} = -\int \frac{dz}{z^2-1} \times \frac{z^2-1}{z^2-2} = -\int \frac{dz}{z^2-2}$$

$$= \frac{1}{2\sqrt2}\left\{ \int \frac{dz}{z+\sqrt2} - \int \frac{dz}{z-\sqrt2} \right\}$$

$$= \frac{1}{2\sqrt2}\log\frac{z+\sqrt2}{z-\sqrt2} = \frac{1}{2\sqrt2}\log\frac{\sqrt{1+x^2}+\sqrt2\,x}{\sqrt{1+x^2}-\sqrt2\,x}$$

$$= \frac{1}{2\sqrt2}\log\frac{(\sqrt{1+x^2}+\sqrt2\,x)^2}{1+x^2-2x^2} = \frac{1}{\sqrt2}\log\frac{\sqrt{1+x^2}+\sqrt2\,x}{\sqrt{1-x^2}}.$$

(35.) $\displaystyle\int \frac{dx}{(1+x)\sqrt{1-x-x^2}}.$ Let $1+x = \frac{1}{z}$

$$x = \frac{1}{z} - 1 \qquad dx = -\frac{dz}{z^2}$$

$$1-x-x^2 = 1 - \frac{1}{z} + 1 - \frac{1}{z^2} + \frac{2}{z} - 1 = 1 + \frac{1}{z} - \frac{1}{z^2},$$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1-x-x^2}} = -\int \frac{dz}{z^2 \dfrac{1}{z} \sqrt{1 + \dfrac{1}{z} - \dfrac{1}{z^2}}}$$

$$= -\int \frac{dz}{\sqrt{z^2 + z - 1}} = -\int \frac{dz}{\sqrt{\left(z + \dfrac{1}{2}\right)^2 - \dfrac{5}{4}}}$$

$$= -\log\left\{\sqrt{z^2 + z - 1} + z + \frac{1}{2}\right\}$$

$$= -\log\left\{\frac{\sqrt{1-x-x^2}}{1+x} + \frac{1}{1+x} + \frac{1}{2}\right\}$$

$$= -\log\left\{\frac{2\sqrt{1-x-x^2} + x + 3}{2(1+x)}\right\}$$

$$= \log\left\{\frac{2(1+x)}{2\sqrt{1-x-x^2} + x + 3}\right\}$$

$$= \log\left\{\frac{2(1+x)(x+3) - 2\sqrt{1-x-x^2}}{x^2 + 6x + 9 - 4 + 4x + 4x^2}\right\}.$$

$$= \log \frac{2(1+x)(x + 3 - 2\sqrt{1-x-x^2})}{5(1+x)^2}$$

$$= \log \frac{x + 3 - 2\sqrt{1-x-x^2}}{1+x} + c.$$

(36.) $$\int \frac{dx}{\sqrt{1 + 2x - x^2}} = \int \frac{dx}{\sqrt{2 - (1-x)^2}}$$

$$= -\int \frac{d(1-x)}{\sqrt{2 - (1-x)^2}} = \cos^{-1} \frac{1-x}{\sqrt{2}}.$$

(37.) $\int \dfrac{dx}{(a + b\,x^2)^{\frac{3}{2}}}$. Let $x = \dfrac{1}{z}, \quad dx = -\dfrac{dz}{z^2}$

$$= -\int \dfrac{dz}{z^2\left(a + \dfrac{b}{z^2}\right)^{\frac{3}{2}}} = -\int \dfrac{z\,dz}{(a\,z^2 + b)^{\frac{3}{2}}}$$

$$= \dfrac{1}{a} \dfrac{1}{(a\,z^2 + b)^{\frac{1}{2}}} = \dfrac{x}{a\,(a + b\,x^2)^{\frac{1}{2}}}.$$

(38.) $\displaystyle\int x\,dx \sqrt{\dfrac{1 + x}{1 - x}} = \int \dfrac{x\,dx}{\sqrt{1 - x^2}} + \int \dfrac{x^2\,dx}{\sqrt{1 - x^2}}$

$$\int x \dfrac{x\,dx}{\sqrt{1 - x^2}} = -x\sqrt{1 - x^2} + \int \sqrt{1 - x^2}\,dx$$

$$= -x\sqrt{1 - x^2} + \int \dfrac{dx}{\sqrt{1 - x^2}}$$

$$-\int \dfrac{x^2\,dx}{\sqrt{1 - x^2}}$$

$$\int x^2 \dfrac{dx}{\sqrt{1 - x^2}} = -\dfrac{x\sqrt{1 - x^2}}{2} + \dfrac{1}{2}\sin^{-1} x,$$

$$\therefore \int x\,dx \sqrt{\dfrac{1 + x}{1 - x}} = -\sqrt{1 - x^2}$$

$$-\dfrac{x}{2}\sqrt{1 - x^2} + \dfrac{1}{2}\sin^{-1} x$$

$$= \dfrac{1}{2}\sin^{-1} x - (2 + x)\dfrac{\sqrt{1 - x^2}}{2}.$$

(39.) $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt{a^3-x^3}}.$ Let $x^3 = z^2$

$$x = z^{\frac{2}{3}},\ dx = \frac{2}{3}\,z^{-\frac{1}{3}}\,dz,\ \sqrt{x} = z^{\frac{1}{3}},$$

$$\therefore \int \frac{\sqrt{x}\,dx}{\sqrt{a^3-x^3}} = \frac{2}{3}\int \frac{dz}{\sqrt{a^3-z^2}} = \frac{2}{3}\sin^{-1}\frac{z}{a^{\frac{3}{2}}}$$

$$= \frac{2}{3}\sin^{-1}\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}$$

Let $\theta = \sin^{-1}\dfrac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}},\ \ \sin\theta = \dfrac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}},$

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{x^{\frac{3}{2}}}{\sqrt{a^3-x^3}},$$

$$\therefore \int \frac{\sqrt{x}\,dx}{\sqrt{a^3-x^3}} = \frac{2}{3}\tan^{-1}\sqrt{\frac{x^3}{a^3-x^3}}.$$

(40.) $\displaystyle\int \frac{dx}{(2ax+x^2)^{\frac{3}{2}}} = \int \frac{dx}{\left((x+a)^2-a^2\right)^{\frac{3}{2}}}.$

Let $x + a = \dfrac{1}{z},\ dx = -\dfrac{dz}{z^2}$

$$= -\int \frac{dz}{z^2\left(\dfrac{1}{z^2}-a^2\right)^{\frac{3}{2}}} = -\int \frac{z\,dz}{(1-a^2z^2)^{\frac{3}{2}}}.$$

$$= -\frac{1}{a^2} \frac{1}{\sqrt{1 - a^2 z^2}} = -\frac{1}{a^2} \frac{1}{\sqrt{1 - \dfrac{a^2}{(x+a)^2}}}$$

$$= -\frac{x + a}{a^2 \sqrt{2 a x + x^2}}.$$

(41.) $\displaystyle\int \frac{d x}{x^2 \sqrt{2 a x - x^2}}.$ Let $x = \dfrac{1}{z}$ $d x = -\dfrac{d z}{z^2}$

$$= -\int \frac{d z}{z^2 \dfrac{1}{z^2} \sqrt{\dfrac{2 a}{z} - \dfrac{1}{z^2}}} = -\int \frac{z \, d z}{\sqrt{2 a z - 1}}.$$

$$= -\frac{1}{a} z \sqrt{2 a z - 1} + \int \sqrt{2 a z - 1} \cdot d z$$

$$= -\frac{1}{a} z \sqrt{2 a z - 1} + \int \frac{2 z \, d z}{\sqrt{2 a z - 1}} - \frac{1}{a} \int \frac{d z}{\sqrt{2 a z - 1}},$$

$$\int \frac{d x}{x^2 \sqrt{2 a x - x^2}} = -\frac{1}{3 a} z \sqrt{2 a z - 1} - \frac{1}{3 a^2} \sqrt{2 a z - 1}.$$

$$= \frac{1}{3 a} \left(\frac{1}{x^2} - \frac{1}{a x} \right) \sqrt{2 a x - x^2}.$$

(42.) $\displaystyle\int x^3 \sqrt{\frac{1 + x^2}{1 - x^2}} d x = \int \frac{x^3 \, d x}{\sqrt{1 - x^4}} + \int \frac{x^5 \, d x}{\sqrt{1 - x^4}},$

$$\int \frac{x^5 \, d x}{\sqrt{1 - x^4}} = \int x^2 \frac{x^3 \, d x}{\sqrt{1 - x^4}}$$

$$= -\frac{x^3}{2} \sqrt{1 - x^4} + \int x \sqrt{1 - x^4} \, d x$$

$$= -\frac{x^2}{2} + \int \frac{x \, d x}{\sqrt{1 - x^4}} - \int \frac{x^5 \, d x}{\sqrt{1 - x^4}},$$

$$\therefore \int \frac{x^5 \, dx}{\sqrt{1 - x^4}} = - \frac{x^2 \sqrt{1 - x^4}}{4} - \frac{1}{4} \int \frac{d \, (- x^2)}{\sqrt{1 - (- x^2)^2}},$$

$$= - \frac{x^2}{4} \sqrt{1 - x^4} + \frac{1}{4} \cos^{-1} (- x^2),$$

$$\therefore \int x^3 \sqrt{\frac{1 + x^2}{1 - x^2}} \, dx$$

$$= - \left(\frac{1}{2} + \frac{x^2}{4} \right) \sqrt{1 - x^4} + - \frac{1}{4} \cos^{-1} (- x^2).$$

Let $\theta = \frac{1}{4} \cos^{-1} (- x^2)$

$$\cos 4 \, \theta = - x^2$$

$$\tan 2 \, \theta = \sqrt{\frac{1 - \cos 4 \, \theta}{1 + \cos 4 \, \theta}} = \sqrt{\frac{1 + x^2}{1 - x^2}}$$

$$\theta = \frac{1}{2} \tan^{-1} \sqrt{\frac{1 + x^2}{1 - x^2}}$$

$$\therefore \int x^3 \sqrt{\frac{1 + x^2}{1 - x^2}} \, dx$$

$$= \frac{1}{2} \tan^{-1} \sqrt{\frac{1 + x^2}{1 - x^2}} - \frac{1}{4} \, (2 + x^2) \sqrt{1 - x^4}.$$

(43.) $\int \frac{d \, x}{x} \sqrt{\frac{a^2 - c^2 x^2}{a^2 - x^2}}.$ Let $y^2 = \frac{a^2 - c^2 x^2}{a^2 - x^2}$

$$a^2 y^2 - x^2 y^2 = a^2 - c^2 x^2$$

$$x^2 = \frac{a^2 (1 - y^2)}{c^2 - y^2},$$

$$2 \, x \, dx = a^2 \left\{ \frac{- 2 y (c^2 - y^2) + 2 y (1 - y^2)}{(c^2 - y^2)^2} \right\} d y$$

$$x\,dx = \frac{a^2\,y\,(1-c^2)\,dy}{(c^2-y^2)^2}$$

$$\frac{dx}{x} = \frac{a^2\,y\,(1-c^2)\,dy}{(c^2-y^2)^2} \cdot \frac{c^2-y^2}{a^2\,(1-y^2)} = \frac{y\,(1-c^2)\,dy}{(1-y^2)\,(c^2-y^2)},$$

$$\therefore \int \frac{dx}{x}\sqrt{\frac{a^2-c^2\,x^2}{a^2-x^2}} \;=\; \int \frac{y^2\,(1-c^2)\,dy}{(1-y^2)\,(c^2-y^2)}$$

$$= -\frac{1}{2}\int\frac{1}{1+y} - \frac{1}{2}\int\frac{1}{1-y} + \frac{c}{2}\int\frac{1}{c+y} + \frac{c}{2}\int\frac{1}{c-y}$$

$$= \frac{1}{2}\log\frac{1-y}{1+y} + \frac{c}{2}\log\frac{c+y}{c-y}$$

$$= \log\sqrt{\left(\frac{1-y}{1+y}\right)\left(\frac{c+y}{c-y}\right)^c}.$$

(44.)
$$\int\frac{x^2+1}{x^2-1}\,\frac{dx}{\sqrt{1-a\,x^2+x^4}}$$

$$= \int\frac{1+x^{-2}}{x-x^{-1}}\,\frac{dx}{\sqrt{x^2-a+x^{-2}}}$$

$$= \int\frac{d\,(x-x^{-1})}{(x-x^{-1})\,\sqrt{(x^{-1}-x)^2-(a-2)}}$$

$$= \frac{1}{\sqrt{a-2}}\sec^{-1}\frac{x^{-1}-x}{\sqrt{a-2}} = \frac{1}{\sqrt{a-2}}\sec^{-1}\frac{x^2-1}{x\sqrt{a-2}}$$

$$= \frac{1}{\sqrt{a-2}}\cos^{-1}\frac{x\sqrt{a-2}}{x^2-1}.$$

FORMULÆ OF REDUCTION.

$$\int \frac{x^n \, dx}{\sqrt{2 \, a \, x - x^2}},$$

$$\int \frac{x^n \, dx}{\sqrt{2 \, a \, x - x^2}} = -\int \frac{x^{n-1} \, (a - x) \, dx}{\sqrt{2 \, a \, x - x^2}} + a \int \frac{x^{n-1} \, dx}{\sqrt{2 \, a \, x - x^2}}.$$

Making $p = x^{n-1}$, $dq = \dfrac{(a - x) \, dx}{\sqrt{2 \, a \, x - x^2}}$,

and $\therefore dp = (n - 1) \, x^{n-2} \, dx$, $q = \sqrt{2 \, a \, x - x^2}$,

we have $\int p \, dq = p q - \int q \, dp$, $\therefore \int \dfrac{x^{n-1} \, (a - x)}{\sqrt{2 \, a \, x - x^2}}$

$$= x^{n-1} \sqrt{2 \, a \, x - x^2} - (n - 1) \int x^{n-2} \sqrt{2 \, a \, x - x^2} \, dx$$

$$= x^{n-1} \sqrt{2 \, a \, x - x^2} - (n - 1) \, . \, 2 \, a \int \frac{x^{n-1} \, dx}{\sqrt{2 \, a \, x - x^2}}$$

$$+ (n - 1) \int \frac{x^n \, dx}{\sqrt{2 \, a \, x - x^2}}.$$

Substituting this value in the original expression, we have

$$\int \frac{x^n \, dx}{\sqrt{2 \, a \, x - x^2}} = - x^{n-1} \sqrt{2 \, a \, x - x^2}$$

$$+ 2 \, a \, . \, (n - 1) \int \frac{x^{n-1} \, dx}{\sqrt{2 \, a \, x - x^2}} - (n - 1) \int \frac{x^n \, dx}{\sqrt{2 \, a \, x - x^2}}$$

$$+ a \int \frac{x^{n-1} \, dx}{\sqrt{2 \, a \, x - x^2}},$$

$$\therefore \left(1+n-1\right)\int \frac{x^n \, dx}{\sqrt{2\,a\,x - x^2}} = -\, x^{n-1}\sqrt{2\,a\,x - x^2}$$

$$+ \, a\,(2\,n-1)\int \frac{x^{n-1}\, dx}{\sqrt{2\,a\,x - x^2}},$$

$$\therefore \int \frac{x^n \, dx}{\sqrt{2\,a\,x - x^2}} = \frac{-x^{n-1}\sqrt{2\,a\,x - x^2}}{n}$$

$$+ \, \frac{a\,(2\,n-1)}{n}\int \frac{x^{n-1}\, dx}{\sqrt{2\,a\,x - x^2}}.$$

$$\int \frac{x^n \, dx}{\sqrt{2\,a\,x + x^2}},$$

$$\int \frac{x^n \, dx}{\sqrt{2\,a\,x + x^2}} = \int x^{n-1}\frac{(a+x)\, dx}{\sqrt{2\,a\,x+x^2}} - a\int \frac{x^{n-1}\, dx}{\sqrt{2\,a\,x + x^2}}$$

$$\int \frac{x^{n-1}(a+x)}{\sqrt{2\,a\,x + x^2}}. \qquad \text{This becomes, if } p = x^{n-1}$$

$$dq = \frac{(a+x)\, dx}{\sqrt{2ax+x^2}}, \; \therefore q = \sqrt{2ax+x^2}, \int p\, dq = pq - \int q\, dp,$$

$$\therefore \int \frac{x^{n-1}(a+x)}{\sqrt{2\,a\,x + x^2}} = x^{n-1}\sqrt{2\,a\,x + x^2}$$

$$-\,(n-1)\int x^{n-2}\sqrt{2\,a\,x + x^2}\, dx \;= x^{n-1}\sqrt{2\,a\,x + x^2}$$

$$-\, 2\,.\,a\,(n-1)\int \frac{x^{n-1}\, dx}{\sqrt{2ax+x^2}} - (n-1)\int \frac{x^n \, dx}{\sqrt{2\,ax+x^2}},$$

Substituting this in the original expression,

$$\int \frac{x^n dx}{\sqrt{2ax+x^2}} = x^{n-1}\sqrt{2ax+x^2} - 2a(n-1)\int \frac{x^{n-1}dx}{\sqrt{2ax+x^2}}$$

$$- (n-1)\int \frac{x^n dx}{\sqrt{2ax+x^2}} - a\int \frac{x^{n-1}dx}{\sqrt{2ax+x^2}}$$

$$= \frac{x^{n-1}\sqrt{2ax+x^2}}{n} - \frac{a(2n-1)}{n}\int \frac{x^{n-1}dx}{\sqrt{2ax+x^2}}.$$

$$\int \frac{dx}{x^n\sqrt{2ax-x^2}}$$

$$\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}} = -\int \frac{dx(a-x)}{x^n\sqrt{2ax-x^2}}$$

$$+ a\int \frac{dx}{x^n\sqrt{2ax-x^2}} \cdots\cdots\cdots (a.)$$

Now, making $p = \dfrac{1}{x^n}$ and $dq = \dfrac{(a-x)dx}{\sqrt{2ax-x^2}}$,

and $\therefore dp = -nx^{-n-1}dx, \quad q = \sqrt{2ax-x^2}$,

there results

$$\int \frac{(a-x)dx}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{x^n} + n\int \frac{\sqrt{2ax-x^2}.dx}{x^{n+1}}$$

$$= \frac{\sqrt{2ax-x^2}}{x^n} + 2an\int \frac{dx}{x^n\sqrt{2ax-x^2}}$$

$$- n\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}}.$$

Substituting this in the equation (a) we obtain

$$\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{x^n} - 2an\int \frac{dx}{x^n\sqrt{2ax-x^2}}$$

$$+ n\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}} + a\int \frac{dx}{x^n\sqrt{2ax-x^2}},$$

$$\therefore\ a(2n-1)\int \frac{dx}{x^n\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{x^n}$$

$$+ (n-1)\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}},$$

$$\therefore \int \frac{dx}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{a(2n-1)x^n}$$

$$+ \frac{n-1}{a(2n-1)}\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}}.$$

$$\int \frac{dx}{x^n\sqrt{2ax+x^2}},$$

$$\int \frac{dx}{x^{n-1}\sqrt{2ax+x^2}} = \int \frac{(a+x)dx}{x^n\sqrt{2ax+x^2}} - a\int \frac{dx}{x^n\sqrt{2ax+x^2}}.$$

If $p = x^{-n}$ and $dq = \dfrac{(a+x)dx}{\sqrt{2ax+x^2}}.$

We have $\displaystyle\int \frac{(a+x)dx}{x^n\sqrt{2ax+x^2}} = \frac{\sqrt{2ax+x^2}}{x^n} + n\int \frac{\sqrt{2ax+x^2}\,dx}{x^{n+1}}$

$$= \frac{\sqrt{2ax+x^2}}{x^n} + 2an\int \frac{dx}{x^n\sqrt{2ax+x^2}},$$

$$+ n\int \frac{dx}{x^{n-1}\sqrt{2ax+x^2}},$$

and by substitution in the first expression there results :

$$\int \frac{dx}{x^{n-1}\sqrt{2ax+x^2}} = \frac{\sqrt{2ax+x^2}}{x^n}.$$

$$+ (2an - a) \int \frac{dx}{x^n \sqrt{2ax + x^2}} + n \int \frac{dx}{x^{n-1} \sqrt{2ax + x^2}},$$

$$\therefore \frac{(1 - n)}{a(2n - 1)} \int \frac{dx}{x^{n-1} \sqrt{2ax + x^2}} - \frac{\sqrt{2ax + x^2}}{x^n a(2n - 1)}$$

$$= \int \frac{dx}{x^n \sqrt{2ax + x^2}}.$$

$$\int \frac{dx}{(a^2 + x^2)^n}. \quad \text{We have } \frac{1}{(a^2 + x^2)^n}$$

$$= \frac{1}{a^2} \cdot \frac{a^2}{(a^2 + x^2)^n} = \frac{1}{a^2} \left\{ \frac{a^2 + x^2}{(a^2 + x^2)^n} - \frac{x^2}{(a^2 + x^2)^n} \right\}$$

$$= \frac{1}{a^2} \cdot \frac{1}{(a^2 + x^2)^{n-1}} - \frac{1}{a^2} \frac{x^2}{(a^2 + x^2)^n}.$$

$$\therefore \int \frac{dx}{(a^2 + x^2)^n} = \frac{1}{a^2} \int \frac{dx}{(a^2 + x^2)^{n-1}} - \frac{1}{a^2} \int \frac{x^2 dx}{(a^2 + x^2)^n}.$$

But $\int \frac{x^2 dx}{(a^2 + x^2)^n} = \int x \frac{x}{(a^2 + x^2)^n} dx = \int p \, dq,$

if $p = x$ and $dq = \frac{x \, dx}{(a^2 + x^2)^n}$;

$$\therefore \int \frac{x^2 dx}{(a^2 + x^2)^n} = \frac{-x}{(2n - 2)(a^2 + x^2)^{n-1}}$$

$$+ \frac{1}{2n - 2} \int \frac{dx}{(a^2 + x^2)^{n-1}};$$

$$\therefore \int \frac{d\,x}{(a^2 + x^2)^n} = \frac{1}{a^2} \frac{x}{(2n-2)\,(a^2 + x^2)^{n-1}}$$

$$- \frac{1}{a^2\cdot(2n-2)} \int \frac{d\,x}{(a^2 + x^2)^{n-1}} + \frac{1}{a^2} \int \frac{d\,x}{(a^2 + x^2)^{n-1}}$$

$$= \frac{1}{a^2} \frac{x}{(2n-2)\,(a^2 + x^2)^{n-1}} + \frac{2n-3}{a^2\,(2n-2)} \int \frac{d\,x}{(a^2 + x^2)^{n-1}}.$$

Also for $\int \dfrac{x^m\,dx}{(a^2 + x^2)^n} = \int x^{m-1}\,dx \dfrac{x}{(a^2 + x^2)^n}$. We have,

if $p = x^{m-1}$, $dq = x\,(a^2 + x^2)^{-n}\,dx$:

$$\int \frac{x^m\,dx}{(a^2 + x^2)^n}$$

$$= \frac{1}{2-2n} \frac{x^{m-1}}{(a^2 + x^2)^{n-1}} + \frac{m-1}{m-2} \int \frac{x^{m-2}\,dx}{(a^2 + x^2)^{n-1}}$$

$$du = (a^2 - x^2)^{\frac{n}{2}}\,dx$$

$$(a^2 - x^2)^{\frac{n}{2}} = a^2\,(a^2 - x^2)^{\frac{n-2}{2}} - x^2\,(a^2 - x^2)^{\frac{n-2}{2}};$$

$$\therefore\ u = a^2 \int (a^2 - x^2)^{\frac{n-2}{2}}\,dx - \int x\,.\,x\,(a^2 - x^2)^{\frac{n-2}{2}}\,dx$$

$$= a^2 \int (a^2 - x^2)^{\frac{n-2}{2}}\,dx + \frac{x\,(a^2 - x^2)^{\frac{n}{2}}}{n} - \frac{u}{n};$$

$$\therefore\ u = \frac{x\,(a^2 - x^2)^{\frac{n}{2}}}{n+1} + \frac{n\,a^2}{n+1} \int (a^2 - x^2)^{\frac{n-2}{2}}\,dx.$$

$$\int \frac{x^n\,dx}{\sqrt{2\,a\,x - x^2}} = -\frac{x^{n-1}\sqrt{2\,a\,x - x^2}}{n}$$

$$+ a\left(\frac{2\,n-1}{n}\right)\int \frac{x^{n-1}\,dx}{\sqrt{2\,a\,x - x^2}}.$$

If we make $n = 2$ this expression becomes

$$\int \frac{x^2\,dx}{\sqrt{2\,a\,x - x^2}} = -\frac{x\sqrt{2\,a\,x - x^2}}{2} + \frac{3\,a}{2}\int \frac{x\,dx}{\sqrt{2\,a\,x - x^2}}.$$

Also $\int \dfrac{x\,dx}{\sqrt{2\,a\,x - x^2}} = -\sqrt{2\,a\,x - x^2} + a\text{ ver sin}^{-1}\left(\dfrac{x}{a}\right)$;

$$\therefore \frac{x^2\,dx}{\sqrt{2\,a\,x - x^2}} = -\frac{x\sqrt{2\,a\,x - x^2}}{2} - \frac{a}{2}\sqrt{2\,a\,x - x^2}$$

$$+ \frac{3\,a^2}{2}\text{ ver sin}^{-1}\left(\frac{x}{a}\right)$$

$$= -\frac{(x + 3\,a)}{2}\sqrt{2\,a\,x - x^2} + \frac{3\,a^2}{2}\text{ ver sin}^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{(a^2 + x^2)^n} = \frac{1}{2\,n - 2}\cdot\frac{x}{a^2\,(a^2 + x^2)^{n-1}} +$$

$$\frac{2\,n - 3}{2\,n - 2}\cdot\frac{1}{a^2}\cdot\int \frac{dx}{(a^2 + x^2)^{n-1}}.$$

If $n = 4$ we have $\int \dfrac{dx}{(a^2 + x^2)^4} =$

$$\frac{1}{6\,a^2}\cdot\frac{x}{(a^2 + x^2)^3} + \frac{5}{6\,a^2}\int \frac{dx}{(a^2 + x^2)^3},$$

$$\int \frac{dx}{(a^2 + x^2)^3} = \frac{1}{4\,a^2} \cdot \frac{x}{(a^2 + x^2)^2} + \frac{3}{4\,a^2} \int \frac{dx}{(a^2 + x^2)^2}.$$

$$\text{Also,} \int \frac{dx}{(a^2 + x^2)^2} = \frac{1}{2} \frac{x}{a^2\,(a^2 + x^2)} + \frac{1}{2} \tan^{-1} \frac{x}{a},$$

$\therefore$ the required integral is

$$\frac{1}{6\,a^2} \cdot \frac{x}{(a^2 + x^2)^3} + \frac{5}{6\,.\,4\,a^4} \frac{x}{(a^2 + x^2)^2}$$

$$+ \frac{5\,.\,3}{6\,.\,4\,.\,2\,a^6} \cdot \frac{x}{a^2 + x^2} + \frac{5\,.\,3}{6\,.\,4\,.\,2} \frac{1}{a^7} \cdot \tan^{-1} \left(\frac{x}{a} \right).$$

$$\int \frac{dx}{x^n\,(1 + x^2)^{\frac{1}{2}}}.$$

The formula of reduction is

$$- \frac{1}{(n - 1)} \frac{(x^2 + 1)^{\frac{1}{2}}}{x^{n-1}} - \frac{n - 2}{n - 1} \int \frac{dx}{x^{n-2}\,(1 + x^2)^{\frac{1}{2}}}.$$

If $n = 6$, we have $\int \dfrac{dx}{x^6\,(1 + x^2)^{\frac{1}{2}}}$

$$= - \frac{1}{5} \cdot \frac{(x^2 + 1)^{\frac{1}{2}}}{x^5} - \frac{4}{5} \cdot \int \frac{dx}{x^4\,(1 + x^2)^{\frac{1}{2}}}$$

$$\int \frac{dx}{x^4\,(1 + x^2)^{\frac{1}{2}}} = - \frac{1}{3} \cdot \frac{(x^2 + 1)^{\frac{1}{2}}}{x^3} - \frac{2}{3} \int \frac{dx}{x^2\,(x^2 + 1)^{\frac{1}{2}}},$$

$$\int \frac{dx}{x^2\,(x^2 + 1)^{\frac{1}{2}}} = \frac{- (1 + x^2)^{\frac{1}{2}}}{x}.$$

And by reduction this becomes

$$\int \frac{dx}{x^6 (x^2 + 1)^{\frac{1}{2}}} = - \frac{1}{5} \frac{(1 + x^2)^{\frac{1}{2}}}{x^5} + \frac{4}{3 \cdot 5} \frac{(1 + x^2)^{\frac{1}{2}}}{x^3}$$

$$- \frac{4 \cdot 2}{5 \cdot 3} \cdot \frac{(1 + x^2)^{\frac{1}{2}}}{x}.$$

$$\int \frac{dx}{x^n \sqrt{(x^2 - 1)}}.$$

The formula of reduction is

$$\int \frac{dx}{x^n (x^2 - 1)^{\frac{1}{2}}} = \frac{1}{n - 1} \cdot \frac{(x^2 - 1)^{\frac{1}{2}}}{x^{n-1}} + \frac{n - 2}{n - 1} \int \frac{dx}{x^{n-2}(x^2 - 1)^{\frac{1}{2}}}.$$

If $n = 6$, this becomes

$$\int \frac{dx}{x^6 (x^2 - 1)^{\frac{1}{2}}} = \frac{1}{5} \cdot \frac{(x^2 - 1)^{\frac{1}{2}}}{x^5} + \frac{4}{5} \int \frac{dx}{x^4 (x^2 - 1)^{\frac{1}{2}}},$$

also $\int \frac{dx}{x^4 (x^2 - 1)^{\frac{1}{2}}} = \frac{1}{3} \cdot \frac{(x^2 - 1)^{\frac{1}{2}}}{x^3} + \frac{2}{3} \int \frac{dx}{x^2 (x^2 - 1)^{\frac{1}{2}}}$

$$\int \frac{dx}{x^2 (x^2 - 1)^{\frac{1}{2}}} = \frac{(x^2 - 1)^{\frac{1}{2}}}{x}.$$

Therefore, we have the required integral

$$= \frac{1}{5} \cdot \frac{(x^2 - 1)^{\frac{1}{2}}}{x^5} + \frac{4}{3 \cdot 5} \frac{(x^2 - 1)^{\frac{1}{2}}}{x^3} + \frac{2 \cdot 4}{3 \cdot 5} \frac{(x^2 - 1)^{\frac{1}{2}}}{x}.$$

CHAPTER IV.

(1.) $\dfrac{d\,u}{d\,x} = x^2(\log x)^2 \qquad (\log x)^2 = p, \qquad \dfrac{d\,q}{d\,x} = x^2$

$$\dfrac{dp}{d\,x} = 2\log x \cdot \dfrac{1}{x}, \quad q = \dfrac{x^3}{3}$$

$$u = \dfrac{x^3(\log x)^2}{3} - \dfrac{2}{3}\int x^2\,dx\,\log x$$

$$\int x^2\,d\,x\,\log x \begin{cases} p = \log x & \dfrac{d\,q}{d\,x} = x^2 \\[2mm] \dfrac{d\,p}{d\,x} = \dfrac{1}{x} & q = \dfrac{x^3}{3} \end{cases}$$

$$= \dfrac{x^3\log x}{3} - \dfrac{1}{3}\int x^2\,d\,x = \dfrac{x^3\log x}{3} - \dfrac{x^3}{9},$$

$$\therefore\ u = \dfrac{x^3(\log x)^2}{3} - \dfrac{2x^3\log x}{9} + \dfrac{2x^3}{27}$$

$$= \dfrac{x^3}{3}\left\{(\log x)^2 - \dfrac{2}{3}\log x + \dfrac{2}{9}\right\}.$$

(2.) $\displaystyle\int \dfrac{x^3\,dx}{\sqrt{\log x}} \quad \begin{cases} p = (\log x)^{-\frac{1}{2}} & dq = x^3\,d\,x \\[2mm] d\,p = -\dfrac{1}{2}(\log x)^{-\frac{3}{2}} \cdot \dfrac{d\,x}{x} & q = \dfrac{x^4}{4} \end{cases}$

$$= \dfrac{x^4}{4}(\log x)^{-\frac{1}{2}} + \dfrac{1}{2 \cdot 4}\int x^3\,d\,x\,(\log x)^{-\frac{3}{2}}$$

Similarly integrating by parts,

$$\int x^3 dx \, (\log x)^{-\frac{3}{2}} = \frac{x^4}{4} (\log x)^{-\frac{3}{2}} + \frac{3}{2 \cdot 4} \int x^3 dx (\log x)^{-\frac{5}{2}},$$

$$\int x^3 dx \, (\log x)^{-\frac{5}{2}} = \frac{x^4}{4} (\log x)^{-\frac{5}{2}} + \frac{5}{2 \cdot 4} \int x^3 dx (\log x)^{-\frac{7}{2}},$$

$$\int x^3 dx (\log x)^{-\frac{7}{2}} = \frac{x^4}{4} (\log x)^{-\frac{7}{2}} + \frac{7}{2 \cdot 4} \int x^3 dx (\log x)^{-\frac{9}{2}},$$

$$\therefore u = \frac{x^4}{4 \sqrt{\log x}}$$

$$+ \frac{1}{8} \left\{ \frac{x^4}{4 (\log x)^3} + \frac{3}{8} \left(\frac{x^4}{4 (\log x)^{\frac{5}{2}}} + \frac{5}{8} \cdot \frac{x^4}{4 (\log x)} \right) \right\},$$

$$u =$$

$$\frac{x^4}{4 \sqrt{\log x}} \left\{ 1 + \frac{1}{8 \log x} + \frac{3}{(8 \log x)^2} + \frac{3 \cdot 5}{(8 \log x)^3} + \&c. \right\}.$$

(3.) $$\int \frac{x^4 dx}{(\log x)^3} = \int x^5 \frac{\dfrac{dx}{x}}{(\log x)^3},$$

$$p = x^5 \qquad dq = \frac{\dfrac{dx}{x}}{(\log x)^3}$$

$$dp = 5 x^4 dx \qquad q = - \frac{1}{2 (\log x)^2},$$

$$\therefore u = - \frac{x^5}{2 (\log x)^2} + \frac{5}{2} \int \frac{x^4 dx}{(\log x)^2}.$$

In like manner,

$$\int \frac{x^4\,dx}{2\,(\log x)^2} = -\frac{x^5}{\log x} + 5\int \frac{x^4\,dx}{\log x},$$

$$\therefore\ u = -\frac{x^5}{2\,(\log x)^2} - \frac{5\,x^5}{2\,\log x} + \frac{25}{2}\int \frac{x^4\,dx}{\log x}.$$

(4.) $\displaystyle\int a^x\,x^3\,dx \begin{cases} p = x^3 & \dfrac{dq}{dx} = a^x \\[2mm] dp = 3\,x^2\,dx & q = \dfrac{a^x}{A} \end{cases}$

$$= \frac{x^3\,a^x}{A} - \frac{3}{A}\int a^x\,x^2\,dx$$

$$\int a^x\,x^2\,dx = \frac{a^x\,x^2}{A} - \frac{2}{A}\int a^x\,x\,dx$$

$$\int a^x\,x\,dx = \frac{a^x\,x}{A} - \frac{1}{A}\int a^x\,dx = \frac{a^x\,x}{A} - \frac{a^x}{A^2},$$

$$\therefore \int a^x\,x^3\,dx = \frac{a^x\,x^3}{A} - \frac{3}{A}\left\{\frac{a^x\,x^2}{A} - \frac{2}{A}\left(\frac{a^x\,x}{A} - \frac{a^x}{A}\right)\right\}$$

$$= a^x\left\{\frac{x^3}{A} - \frac{3\,x^2}{A^2} + \frac{6\,x}{A^3} - \frac{6}{A^4}\right\}.$$

(5.) $\displaystyle\int x^3\,(\log x)^3\,dx \quad p = (\log x)^3 \quad dq = x^3\,dx$

$$dp = 3\,(\log x)^2\,\frac{dx}{x} \qquad q = \frac{x^4}{4}$$

$$\int x^3\,(\log x)^3\,dx = \frac{x^4}{4}\,(\log x)^3 - \frac{3}{4}\int x^3\,(\log x)^2\,dx$$

similarly, $\displaystyle\int x^3\,(\log x)^2\,dx = \frac{x^4}{4}\,(\log x)^2 - \frac{2}{4}\int x^3\,\log x\,dx$

$$\int x^3 \, dx \log x = \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{16} x^4,$$

$$\therefore \int x^3 (\log x)^3 \, dx =$$

$$\frac{x^4}{4} \left\{ (\log x)^3 - \frac{3}{4} (\log x)^2 + \frac{3 \cdot 2}{4^2} \log x - \frac{3 \cdot 2}{4^3} \right\}.$$

(6.) $\int e^x x^4 \, dx = e^x x^4 - 4 \int e^x x^3 \, dx$

$$\int e^x x^3 \, dx = e^x x^3 - 3 \int e^x x^2 \, dx$$

$$\int e^x x^2 \, dx = e^x x^2 - 2 \int e^x x \, dx$$

$$\int e^x x \, dx = e^x x - \int e^x \, dx = e^x x - e^x,$$

$$\therefore \int e^x x^4 \, dx = e^x (x^4 - 4 x^3 + 12 x^2 - 24 x + 24).$$

(7.) $\int e^{-x} x^3 \, dx = - e^{-x} x^3 + 3 \int e^{-x} x^2 \, dx$

$$\int e^{-x} x^2 \, dx = - e^{-x} x^2 + 2 \int e^{-x} x \, dx$$

$$\int e^{-x} x \, dx = - e^{-x} x + \int e^{-x} \, dx = - e^{-x} x - e^{-x},$$

$$\therefore \int e^{-x} x^3 \, dx = - e^{-x} (x^3 + 3 x^2 + 6 x + 6).$$

(8.) $\int \dfrac{x\,dx}{\sqrt{1+x^2}} \log x$ $\quad \left\{ \begin{array}{ll} p = \log x & \dfrac{dq}{dx} = \dfrac{x}{\sqrt{1+x^2}} \\[2ex] \dfrac{dp}{dx} = \dfrac{1}{x} & q = \sqrt{1+x^2} \end{array} \right\}$

$$= \sqrt{1+x^2}\,\log x - \int \frac{\sqrt{1+x^2}\,dx}{x}$$

$$= \sqrt{1+x^2}\,\log x - \int \frac{dx}{x\sqrt{1+x^2}} - \int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$= \sqrt{1+x^2}\,\log x - \log \frac{x}{1+\sqrt{1+x^2}} - \sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\,(\log x - 1) - \log \frac{x}{1+\sqrt{1+x^2}}$$

$$= \sqrt{1+x^2}\,\log\left(\frac{x}{e}\right) - \log \frac{x}{1+\sqrt{1+x^2}}.$$

(9.) $\displaystyle\int e^x\,dx\,\frac{x^2+1}{(x+1)^2} = \int e^x\,dx\,\frac{x^2-1+2}{(x+1)^2}$

$$= \int e^x\,dx \left\{ \frac{x-1}{x+1} + \frac{(x+1)-(x-1)}{(x+1)^2} \right\}$$

$$= \int e^x \left\{ \frac{(x-1)}{x+1}\,dx + d\left(\frac{x-1}{x+1}\right) \right\}$$

$$= e^x\,\frac{x-1}{x+1}.$$

(10.) $\displaystyle\int e^x \frac{(2-x^2)dx}{(1-x)\sqrt{1-x^2}} = \int e^x\, dx\,\frac{1-x^2+1}{(1-x)\sqrt{1-x^2}}$

$\displaystyle = \int e^x\, dx\left\{\frac{\sqrt{1+x}}{\sqrt{1-x}} + \frac{1}{(1-x)\sqrt{1-x^2}}\right\},$

$\displaystyle = \int e^x\, dx\left\{\frac{\sqrt{1+x}}{\sqrt{1-x}} + \frac{(1-x)+(1+x)}{(1-x)^2}\cdot\frac{1}{2}\,\frac{\sqrt{1-x}}{\sqrt{1+x}}\right\}.$

$\displaystyle = \int e^x\left\{\frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx + d\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)\right\}$

$\displaystyle = e^x\,\frac{\sqrt{1+x}}{\sqrt{1-x}}.$

(11.) $\displaystyle\int \frac{xe^x\,dx}{(e^x-1)^3} \qquad p=x \qquad dq=\frac{e^x\,dx}{(e^x-1)^3}$

$\displaystyle \qquad\qquad\qquad\qquad dp=dx \qquad q=-\frac{dx}{2(e^x-1)^2},$

$\displaystyle \int \frac{xe^x\,dx}{(e^x-1)^3} = -\frac{x}{2(e^x-1)^2} + \frac{1}{2}\int\frac{dx}{(e^x-1)^2}$

$\displaystyle \qquad = -\frac{x}{2(e^x-1)^2} - \frac{1}{2}\int\frac{dx}{e^x-1} + \frac{1}{2}\int\frac{e^x\,dx}{(e^x-1)^2}$

$\displaystyle = -\frac{x}{2(e^x-1)^2} + \frac{1}{2}\int\frac{e^x\,dx}{e^x} - \frac{1}{2}\int\frac{e^x\,dx}{e^x-1} + \frac{1}{2}\int\frac{e^x\,dx}{(e^x-1)^2}$

$\displaystyle \qquad = -\frac{x}{2(e^x-1)^2} + \frac{1}{2}\,\log\frac{e^x}{e^x-1} - \frac{1}{2}\,\frac{1}{e^x-1}.$

$\displaystyle \qquad = \frac{1}{2}\,\log\frac{e^x}{e^x-1} - \frac{1}{2(e^x-1)}\left\{1+\frac{x}{e^x-1}\right\}.$

(12.) $\dfrac{du}{dx} = \dfrac{a^x}{x^4} \qquad p = a^x \qquad \dfrac{dq}{dx} = \dfrac{1}{x^4},$

$$\dfrac{dp}{dx} = \mathrm{A}\,a^x \qquad q = -\dfrac{1}{3\,x^3},$$

$$\therefore u = -\dfrac{a^x}{3\,x^3} + \dfrac{\mathrm{A}}{3}\int \dfrac{a^x\,dx}{x^3}.$$

Similarly,

$$\int \dfrac{a^x\,dx}{x^3} = -\dfrac{a^x}{2\,x^2} + \dfrac{\mathrm{A}}{2}\int \dfrac{a^x\,dx}{x^2}$$

$$\int \dfrac{a^x\,dx}{x^2} = -\dfrac{a^x}{x} + \mathrm{A}\int \dfrac{a^x\,dx}{x},$$

$$\therefore u = -\dfrac{a^x}{3\,x^3} + \dfrac{\mathrm{A}}{3}\left\{ -\dfrac{a^x}{2\,x^2} + \dfrac{\mathrm{A}}{2}\left(-\dfrac{a^x}{x} + \mathrm{A}\int \dfrac{a^x\,dx}{x} \right) \right\}$$

$$= -a^x\left\{ \dfrac{1}{3\,x^3} + \dfrac{\mathrm{A}}{2\,.\,3\,x^2} + \dfrac{\mathrm{A}^2}{2\,.\,3\,x} \right\} + \dfrac{\mathrm{A}^3}{2\,.\,3}\int \dfrac{a^x\,dx}{x}.$$

(13.) $\dfrac{du}{dx} = \dfrac{a^x}{\sqrt{x}} \qquad p = x^{-\frac{1}{2}} \qquad \dfrac{dq}{dx} = a^x,$

$$\dfrac{dp}{dx} = -\dfrac{1}{2}x^{-\frac{3}{2}} \quad q = \dfrac{a^x}{\mathrm{A}},$$

$$\therefore u = \dfrac{a^x}{\mathrm{A}\,x^{\frac{1}{2}}} + \dfrac{1}{2\,\mathrm{A}}\int x^{-\frac{3}{2}}a^x\,dx,$$

also $\displaystyle\int x^{-\frac{3}{2}}a^x\,dx = \dfrac{a^x}{\mathrm{A}\,x^{\frac{3}{2}}} + \dfrac{3}{2\,\mathrm{A}}\int x^{-\frac{5}{2}}a^x\,dx$

$$\int x^{-\frac{5}{2}}a^x\,dx = \dfrac{a^x}{\mathrm{A}\,x^{\frac{5}{2}}} + \dfrac{5}{2\,\mathrm{A}}\int x^{-\frac{7}{2}}a^x\,dx,$$

$$\therefore u = \frac{a^x}{A\,x^{\frac{1}{2}}} + \frac{1}{2\,A}\left\{\frac{a^x}{A\,x^{\frac{3}{2}}} + \frac{3}{2\,A}\left(\frac{a^x}{A\,x^{\frac{5}{2}}} + \frac{5}{2\,A}\frac{a^x}{A\,x^{\frac{7}{2}}}\&c.\right)\right\}$$

$$= \frac{a^x}{A\,\sqrt{x}}\left\{1 + \frac{1}{2x\,A} + \frac{3}{(2\,x\,A)^2} + \frac{3\,.\,5}{(2\,x\,A)^3},\&c.\right\}.$$

Another result may be obtained by the following method.

$$\frac{d\,u}{d\,x} = \frac{a^x}{\sqrt{x}} \qquad p = a^x \qquad \frac{d\,q}{d\,x} = x^{-\frac{1}{2}}$$

$$\frac{d\,p}{d\,x} = A\,a^x \qquad q = 2\,x^{\frac{1}{2}},$$

$$\therefore u = 2\,a^x\,x^{\frac{1}{2}} - 2\,A\int a^x\,x^{\frac{1}{2}}\,dx.$$

Also, $\displaystyle\int a^x\,x^{\frac{1}{2}}\,dx = \frac{2}{3}\,a^x\,x^{\frac{3}{2}} - \frac{2}{3}\,A\int a^x\,x^{\frac{3}{2}}\,dx.$

$$\int a^x\,x^{\frac{3}{2}}\,dx = \frac{2}{5}\,a^x\,x^{\frac{5}{2}} - \frac{2}{5}\,A\int a^x\,x^{\frac{5}{2}}\,dx,$$

$$\int a^x\,x^{\frac{5}{2}}\,dx = \frac{2}{7}\,a^x\,x^{\frac{7}{2}} - \frac{2}{7}\,A\int a^x\,x^{\frac{7}{2}}\,dx,$$

$$\therefore u = 2\,a^x\,x^{\frac{1}{2}} - 2\,A\left\{\frac{2}{3}\,a^x\,x^{\frac{3}{2}} - \frac{2}{3}\,A\left(\frac{2}{5}\,a^x\,x^{\frac{5}{2}}\right.\right.$$

$$\left.\left. - \frac{2}{5}\,A\,\frac{2}{7}\,a^x\,x^{\frac{7}{2}},\&c.\right)\right\},$$

$$\therefore u = \frac{a^x}{A\,\sqrt{x}}\left\{2\,x\,A - \frac{(2x\,A)^2}{3} + \frac{(2\,x\,A)^3}{3\,.\,5} - \frac{(2\,x\,A)^4}{3\,.\,5\,.\,7}\right.$$

$$\left. + \&c.\right\}.$$

(14.) $\quad \dfrac{d\,u}{d\,x} = x^{n\,x}x^m = x^m \left\{ 1 + n\,x\,\log x \right.$

$$\left. + \frac{(n\,x\,\log x)^2}{1\,.\,2} + \&c. \right\}$$

$$= x^m + n\,x^{m+1}\,\log x + \frac{n^2}{2}\,x^{m+2}\,(\log x)^2 + \&c.,$$

$$\therefore u = \frac{x^{m+1}}{m+1} + n\int x^{m+1}\,\log x\,d\,x$$

$$+ \frac{n^2}{1\,.\,2}\int x^{m+2}\,(\log x)^2\,d\,x + \&c.$$

But $\displaystyle\int x^m(\log x)^n\,dx = \frac{x^{m+1}}{m+1}\left\{ (\log x)^n - \frac{n}{m+1}(\log x)^{n-1} \right.$

$$+ \frac{n\,(n-1)}{(m+1)^2}\,(\log x)^{n-2} - \&c.$$

$$\pm \frac{n\,(n-1)\,(n-2)\,\ldots\,2\,.\,1}{(m+1)^{n+1}}\,x^{m+1} \left. \right\},$$

$$\therefore \int x^{m+1}\log x\,dx = \frac{x^{m+2}}{m+2}\left\{ \log x - \frac{1}{m+2} \right\}$$

$$\int x^{m+2}(\log x)^2\,dx = \frac{x^{m+3}}{m+3}\left\{ (\log x)^2 \right.$$

$$- \frac{2}{m+3}\,\log.x + \frac{2}{(m+3)^2} \left. \right\},$$

$$\int x^{m+3}\,(\log x)^3\,dx = \frac{x^{m+4}}{m+4}\left\{ (\log x)^3 - \frac{3}{m+4}\,(\log x)^2 \right.$$

$$+ \frac{6}{(m+4)^2}\,\log x - \frac{6}{(m+4)^3} \left. \right\}$$

Arranging the terms according to the powers of $\log x$,

$$\int x^{nx} \cdot x^m \, dx = \frac{x^{m+1}}{m+1} - \frac{n \, x^{m+2}}{(m+2)^2} + \frac{n^2 \, x^{m+3}}{(m+3)^3} - \frac{n^3 \, x^{m+4}}{(m+4)^4} + \&c.$$

$$+ \, n \log x \left\{ \frac{x^{n+2}}{m+2} - \frac{n x^{m+3}}{(m+3)} + \frac{n^2 x^{m+4}}{(m+4)^i}, \, \&c. \right\}$$

$$+ \, \frac{n^2}{1 \cdot 2} (\log x)^2 \left\{ \frac{x^{m+3}}{m+3} - \frac{n \, x^{m+4}}{(m+4)^2} + \&c. \right\}.$$

If $x = 0$ all the terms vanish.

If $x = 1, \int x^{nx} \cdot x^m \, dx$ becomes

$$\frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{n^2}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \&c.$$

(15.) $\dfrac{d u}{d x} = \dfrac{\log x}{(1+x)^2} \qquad p = \log x \qquad \dfrac{d q}{d x} = \dfrac{1}{(1+x)^2}$

$$\frac{d p}{d x} = \frac{1}{x} \qquad q = - \frac{1}{1+x},$$

$$\therefore \, u = - \frac{\log x}{1+x} + \int \frac{d x}{x \, (1+x)}$$

$$= - \frac{\log x}{1+x} + \int \frac{d x}{x} - \int \frac{d x}{1+x}$$

$$= - \frac{\log x}{1+x} + \log x - \log (1+x)$$

$$= \frac{x}{1+x} \log x - \log (1+x).$$

CHAPTER V.

(1.) $\displaystyle\int \sin^3 \theta\, d\theta = \int \sin \theta\, (1 - \cos^2 \theta)\, d\theta$

$$= \int \sin\theta\, d\theta - \int \sin\theta \cos^2\theta\, d\theta$$

$$= -\cos\theta + \frac{\cos^3 \theta}{3} = -\cos\theta + \frac{\cos\theta\,(1 - \sin^2 \theta)}{3}$$

$$= -\frac{2}{3}\cos\theta - \frac{\cos\theta \sin^2 \theta}{3}$$

(2.) $\displaystyle\int \cos^3 \theta\, d\theta = \int \cos\theta\, (1 - \sin^2 \theta)\, d\theta$

$$= \int \cos\theta\, d\theta - \int \cos\theta \sin^2 \theta\, d\theta$$

$$= \sin\theta - \frac{\sin^3 \theta}{3} = \sin\theta - \frac{\sin\theta\,(1 - \cos^2 \theta)}{3}$$

$$= \frac{2 \sin\theta}{3} + \frac{\sin\theta \cos^2 \theta}{3}.$$

(3.) $\int \dfrac{d\theta}{\sin^3 \theta} = \int \dfrac{(\sin^2 \theta + \cos^2 \theta)\, d\theta}{\sin^3 \theta}$

$$= \int \dfrac{d\theta}{\sin \theta} + \int \dfrac{\cos^2 \theta\, d\theta}{\sin^3 \theta}$$

$$\int \cos \theta\, \dfrac{\cos \theta\, d\theta}{\sin^3 \theta} = - \cos \theta\, \dfrac{1}{2 \sin^2 \theta} - \dfrac{1}{2} \int \dfrac{\sin \theta\, d\theta}{\sin^2 \theta},$$

$$\therefore \int \dfrac{d\theta}{\sin^3 \theta} = \dfrac{1}{2} \int \dfrac{d\theta}{\sin \theta} - \dfrac{1}{2} \dfrac{\cos \theta}{\sin^2 \theta}$$

$$= \tfrac{1}{2} \log \left(\tan \dfrac{\theta}{2} \right) - \dfrac{1}{2} \dfrac{\cos \theta}{\sin^2 \theta}.$$

(4.) $\int \dfrac{\sin^3 \theta\, d\theta}{\cos^4 \theta} = \int \dfrac{\sin \theta\, d\theta}{\cos^4 \theta} - \int \dfrac{\sin \theta\, d\theta}{\cos^2 \theta}$

$$= \dfrac{1}{3 \cos^3 \theta} - \dfrac{1}{\cos \theta} = \dfrac{1}{3 \cos^3 \theta} \left\{ \dfrac{1}{3} - 1 + \sin^2 \theta \right\}$$

$$= \dfrac{1}{3 \cos^3 \theta} \left\{ \sin^2 \theta - \dfrac{2}{3} \right\}.$$

(5.) $\int x^2 \tan^{-1} \sqrt{\dfrac{x}{a}}\, dx.$ Let $\dfrac{x}{a} = z^2$

$$x^3 = a^3 z^6 \qquad\qquad x^2\, dx = 2\, a^3 z^5\, dz,$$

$$\therefore \int x^2 \tan^{-1} \sqrt{\dfrac{x}{a}}\, dx = 2\, a^3 \int z^5 \tan^{-1} z\, dz$$

$$p = \tan^{-1} z \qquad\qquad dq = 2\, a^3 z^5\, dz$$

$$dp = \dfrac{1}{1 + z^2}\, dz \qquad\qquad q = \dfrac{a^3 z^6}{3},$$

$$\therefore u = \frac{a^3 z^6}{3} \tan^{-1} z - \frac{a^3}{3} \int \frac{z^6}{1 + z^2} dz$$

$$= \frac{a^3 z^6}{3} \tan^{-1} z - \frac{a^3}{3} \int \left\{ z^4 - z^2 + 1 - \frac{1}{1 + z^2} \right\} dz$$

$$= \frac{a^3 z^6}{3} \tan^{-1} z - \frac{a^3}{3} \left\{ \frac{z^5}{5} - \frac{z^3}{3} + z \right\} + \frac{a^3}{3} \tan^{-1} z$$

$$= \frac{a^3}{3} (z^6 + 1) \tan^{-1} z - \frac{a^3 z}{3} \left\{ \frac{z^4}{5} - \frac{z^2}{3} + 1 \right\}$$

$$= \frac{a^3}{3} \left(\frac{x^3 + a^3}{a^3} \right) \tan^{-1} \sqrt{\frac{x}{a}} - \frac{a^3 \sqrt{x}}{3 \sqrt{a}} \left\{ \frac{x^2}{5 a^2} - \frac{x}{3 a} + 1 \right\}$$

$$= \frac{x^3 + a^3}{3} \tan^{-1} \sqrt{\frac{x}{a}} - \frac{\sqrt{a x}}{3} \left(\frac{x^2}{5} - \frac{a x}{3} + a^2 \right).$$

(6.) $\quad \dfrac{d u}{d \theta} = \cos^6 \theta \qquad p = \cos^5 \theta \qquad \dfrac{d q}{d \theta} = \cos \theta$

$$\frac{d p}{d \theta} = - 5 \cos^4 \theta \sin \theta \quad q = \sin \theta,$$

$$\therefore \int \cos^6 \theta \, d \theta = \sin \theta \cos^5 \theta + 5 \int \cos^4 \theta \sin^2 \theta \, d \theta$$

$$= \sin \theta \cos^5 \theta + 5 \int \cos^4 \theta \, dx - 5 \int \cos^6 \theta \, dx,$$

$$\therefore \int \cos^6 \theta \, d \theta = \frac{\sin \theta \cos^5 \theta}{6} + \frac{5}{6} \int \cos^4 \theta \, d \theta.$$

Similarly,

$$\int \cos^4 \theta \, d \theta = \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{4} \int \cos^2 \theta \, d \theta$$

$$\int \cos^2 \theta \, d\theta = \frac{1}{2} \sin \theta \, \cos \theta + \frac{1}{2} \int d\theta = \frac{1}{2} \sin \theta \cos \theta + \frac{\theta}{2},$$

$$\therefore \int \cos^6 \theta \, d\theta = \frac{\sin \theta \cos^5 \theta}{6} + \frac{5}{6} \left\{ \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{4} \right.$$

$$\left. \left(\frac{1}{2} \sin \theta \cos \theta + \frac{\theta}{2} \right) \right\}$$

$$= \sin \theta \left\{ \frac{\cos^5 \theta}{6} + \frac{5 \cos^3 \theta}{24} + \frac{5 \cos \theta}{16} \right\} + \frac{5}{16} \frac{\theta}{}.$$

(7.) $\dfrac{d u}{d \theta} = \sin^2 \theta \cos^4 \theta = \cos^4 \theta - \cos^6 \theta.$

But from preceding example,

$$\int \cos^4 \theta \, d\theta = \sin \theta \left\{ \frac{\cos^3 \theta}{4} + \frac{3 \cos \theta}{8} \right\} + \frac{3 \theta}{8}$$

$$\int \cos^6 \theta \, d\theta = \sin \theta \left\{ \frac{\cos^5 \theta}{6} + \frac{5 \cos^3 \theta}{24} + \frac{5 \cos \theta}{16} \right\} + \frac{5 \theta}{16}$$

$$\therefore u = \sin \theta \left\{ - \frac{\cos^5 \theta}{6} + \frac{\cos^3 \theta}{24} + \frac{\cos \theta}{16} \right\} + \frac{\theta}{16}$$

$$= \sin \theta \left\{ - \frac{\cos^3 \theta}{6} + \frac{\sin^2 \theta \cos^3 \theta}{6} + \frac{\cos^3 \theta}{24} + \frac{\cos \theta}{16} \right\} + \frac{\theta}{16}$$

$$= \sin \theta \left\{ \frac{\sin^2 \theta \cos^3 \theta}{6} - \frac{1}{8} \cos \theta \left(1 - \sin^2 \theta \right) + \frac{\cos \theta}{16} \right\} + \frac{\theta}{16}$$

$$= \frac{\sin^3 \theta \cos^3 \theta}{6} + \frac{\sin^3 \theta \cos \theta}{8} - \frac{\sin \theta \cos \theta}{16} + \frac{\theta}{16}.$$

(8.) $\dfrac{du}{d\theta} = \sin^6\theta \cos^3\theta = \sin^6\theta \cos\theta - \sin^8\theta \cos\theta,$

$$u = \frac{\sin^7\theta}{7} - \frac{\sin^9\theta}{9} = \sin^7\theta \left\{ \frac{1}{7} - \frac{\sin^2\theta}{9} \right\}$$

$$= \sin^7\theta \left\{ \frac{1}{7} - \frac{1 - \cos^2\theta}{9} \right\}$$

$$= \sin^7\theta \left\{ \frac{\cos^2\theta}{9} + \frac{2}{63} \right\}.$$

(9.) $\displaystyle\int \frac{d\theta}{\sin^5\theta} = \int \frac{\cos^2\theta\, d\theta}{\sin^5\theta} + \int \frac{d\theta}{\sin^3\theta},$

$$\int \frac{\cos^2\theta\, d\theta}{\sin^5\theta} \begin{cases} p = \cos\theta & \dfrac{dq}{d\theta} = \dfrac{\cos\theta}{\sin^5\theta} \\[2mm] \dfrac{dp}{d\theta} = -\sin\theta & q = -\dfrac{1}{4\sin^4\theta} \end{cases}$$

$$\int \frac{\cos^2\theta\, d\theta}{\sin^5\theta} = -\frac{\cos\theta}{4\sin^4\theta} - \frac{1}{4}\int \frac{d\theta}{\sin^3\theta},$$

$$\therefore\ u = -\frac{\cos\theta}{4\sin^4\theta} + \frac{3}{4}\int \frac{d\theta}{\sin^3\theta},$$

$$\int \frac{d\theta}{\sin^3\theta} = -\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{2}\int \frac{d\theta}{\sin\theta}.$$

$$\int \frac{d\theta}{\sin^3\theta} = -\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{2}\log\left(\tan\frac{\theta}{2}\right),$$

$$\therefore\ u = -\frac{\cos\theta}{4\sin^4\theta} - \frac{3\cos\theta}{8\sin^2\theta} + \frac{3}{8}\log\left(\tan\frac{\theta}{2}\right)$$

$$= -\cos\theta\left\{ \frac{1}{4\sin^4\theta} + \frac{3}{8\sin^2\theta} \right\} + \frac{3}{8}\log\left(\tan\frac{\theta}{2}\right).$$

(10.) $\displaystyle\int_{\iota} \frac{d\theta}{\cos^6\theta} = \int \frac{\sin^2\theta\, d\theta}{\cos^6\theta} + \int \frac{d\theta}{\cos^4\theta}$

$$p = \sin\theta \qquad \frac{dq}{d\theta} = \frac{\sin\theta}{\cos^6\theta},$$

$$\frac{dp}{d\theta} = \cos\theta \qquad q = \frac{1}{5\cos^5\theta},$$

$$\therefore \int \frac{\sin^2\theta\, d\theta}{\cos^6\theta} = \frac{\sin\theta}{5\cos^5\theta} - \frac{1}{5}\int \frac{d\theta}{\cos^4\theta},$$

$$\therefore u = \frac{\sin\theta}{5\cos^5\theta} + \frac{4}{5}\int \frac{d\theta}{\cos^4\theta}.$$

Similarly,

$$\int \frac{d\theta}{\cos^4\theta} = \frac{\sin\theta}{3\cos^3\theta} + \frac{2}{3}\int \frac{d\theta}{\cos^2\theta}$$

$$= \frac{\sin\theta}{3\cos^3\theta} + \frac{2}{3}\frac{\sin\theta}{\cos\theta},$$

$$\therefore u = \frac{\sin\theta}{5\cos^5\theta} + \frac{4}{5}\left\{ \frac{\sin\theta}{3\cos^3\theta} + \frac{2}{3}\frac{\sin\theta}{\cos\theta}\right\}$$

$$= \sin\theta\left\{ \frac{1}{5\cos^5\theta} + \frac{4}{15\cos^3\theta} + \frac{8}{15\cos\theta}\right\}.$$

(11.) $\displaystyle\frac{du}{d\theta} = \frac{\sin^5\theta}{\cos^2\theta} = \frac{\sin\theta\,(1 - 2\cos^2\theta + \cos^4\theta)}{\cos^2\theta}$

$$= \frac{\sin\theta}{\cos^2\theta} - 2\sin\theta + \sin\theta\cos^2\theta,$$

$$\therefore u = \frac{1}{\cos\theta} + 2\cos\theta - \frac{\cos^3\theta}{3}$$

$$= \frac{1}{\cos\theta}\left\{1 + 2\cos^2\theta - \frac{\cos^4\theta}{3}\right\}$$

$$= \frac{1}{\cos\theta}\left\{1 + 2 - 2\sin^2\theta - \frac{1 - 2\sin^2\theta + \sin^4\theta}{3}\right\}$$

$$= -\frac{1}{\cos\theta}\left\{\frac{\sin^4\theta}{3} + \frac{4\sin^2\theta}{3} - \frac{8}{3}\right\}.$$

(12.) $\dfrac{du}{d\theta} = \dfrac{\cos^4\theta}{\sin^3\theta} \qquad p = \cos^3\theta \qquad\qquad \dfrac{dq}{d\theta} = \dfrac{\cos\theta}{\sin^3\theta}$

$$\frac{dp}{d\theta} = -3\sin\theta\cos^2\theta \qquad q = -\frac{1}{2\sin^2\theta},$$

$$\therefore\ u = -\frac{\cos^3\theta}{2\sin^2\theta} - \frac{3}{2}\int\frac{\cos^2\theta\, d\theta}{\sin\theta}$$

$$= -\frac{\cos^3\theta}{2\sin^2\theta} - \frac{3}{2}\int\frac{\cos^2\theta\, d\theta}{\sin^3\theta} + \frac{3}{2}\int\frac{\cos^4\theta\, d\theta}{\sin^3\theta},$$

$$\therefore\ u = \frac{\cos^3\theta}{\sin^2\theta} + 3\int\frac{\cos^2\theta\, d\theta}{\sin^3\theta}$$

$$p = \cos\theta \qquad\qquad \frac{dq}{d\theta} = \frac{\cos\theta}{\sin^3\theta}$$

$$\frac{dp}{d\theta} = -\sin\theta \qquad\qquad q = -\frac{1}{2\sin^2\theta}$$

$$\int\frac{\cos^2\theta\, d\theta}{\sin^3\theta} = -\frac{\cos\theta}{2\sin^2\theta} - \frac{1}{2}\int\frac{d\theta}{\sin\theta}$$

$$= -\frac{\cos\theta}{2\sin^2\theta} - \frac{1}{2}\log\left(\tan\frac{\theta}{2}\right),$$

$$\therefore \; u = \frac{\cos^3 \theta}{\sin^2 \theta} - \frac{3 \cos \theta}{2 \sin^2 \theta} - \frac{3}{2} \log \left(\tan \frac{\theta}{2} \right)$$

$$= \frac{1}{\sin^2 \theta} \left\{ \cos^3 \theta - \frac{3 \cos \theta}{2} \right\} - \frac{3}{2} \log \left(\tan \frac{\theta}{2} \right).$$

(13.) $\dfrac{du}{d\theta} = \dfrac{1}{\sin^2 \theta \cos^3 \theta} = \dfrac{1}{\cos^3 \theta} + \dfrac{1}{\sin^2 \theta \cos \theta}$

$$= \frac{\sin^2 \theta}{\cos^3 \theta} + \frac{1}{\cos \theta} + \frac{1}{\cos \theta} + \frac{\cos \theta}{\sin^2 \theta},$$

$$u = \int \frac{\sin^2 \theta \, d\theta}{\cos^3 \theta} + 2 \int \frac{d\theta}{\cos \theta} - \frac{1}{\sin \theta}$$

$$p = \sin \theta \qquad\qquad \frac{dq}{d\theta} = \frac{\sin \theta}{\cos^3 \theta}$$

$$\frac{dp}{d\theta} = \cos \theta \qquad\qquad q = \frac{1}{2 \cos^2 \theta},$$

$$\therefore \int \frac{\sin^2 \theta \, d\theta}{\cos^3 \theta} = \frac{\sin \theta}{2 \cos^2 \theta} - \frac{1}{2} \int \frac{d\theta}{\cos \theta},$$

$$\therefore \; u = \frac{\sin \theta}{2 \cos^2 \theta} + \frac{3}{2} \int \frac{d\theta}{\cos \theta} - \frac{1}{\sin \theta}$$

$$= \frac{1}{\sin \theta} \left\{ \frac{1 - \cos^2 \theta}{2 \cos^2 \theta} - 1 \right\} + \frac{3}{2} \log \left\{ \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right\}$$

$$= \frac{1}{\sin \theta} \left\{ \frac{1}{2 \cos^2 \theta} - \frac{3}{2} \right\} + \frac{3}{2} \log \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right).$$

(14.) $\quad \dfrac{du}{d\theta} = \dfrac{1}{\sin^4\theta\cos^2\theta} = \dfrac{1}{\sin^2\theta\cos^2\theta} + \dfrac{1}{\sin^4\theta}$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} + \frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^4\theta},$$

$$\therefore \; u = \tan\theta + 2\int\frac{d\theta}{\sin^2\theta} + \int\frac{\cos^2\theta\,d\theta}{\sin^4\theta}$$

$$\int\frac{\cos^2\theta\,d\theta}{\sin^4\theta}\begin{cases} p = \cos\theta & \dfrac{dq}{d\theta} = \dfrac{\cos\theta}{\sin^4\theta} \\[2mm] \dfrac{dp}{d\theta} = -\sin\theta & q = -\dfrac{1}{3\sin^3\theta} \end{cases}$$

$$= -\frac{\cos\theta}{3\sin^3\theta} - \frac{1}{3}\int\frac{d\theta}{\sin^2\theta},$$

$$\therefore \; u = -\frac{\cos\theta}{3\sin^3\theta} + \tan\theta + \frac{5}{3}\int\frac{d\theta}{\sin^2\theta}$$

$$= -\frac{\cos^2\theta}{3\sin^3\theta\cos\theta} + \frac{\sin\theta}{\cos\theta} - \frac{5}{3}\frac{\cos\theta}{\sin\theta}$$

$$= -\frac{1-\sin^2\theta}{3\sin^3\theta\cos\theta} + \frac{3\sin^2\theta - 5\cos^2\theta}{3\sin\theta\cos\theta}$$

$$= -\frac{1}{3\sin^3\theta\cos\theta} + \frac{1+3\sin^2\theta - 5\cos^2\theta}{3\sin\theta\cos\theta}.$$

$$= -\frac{1}{3\sin^3\theta\cos\theta} + \frac{4\,(\sin^2\theta - \cos^2\theta)}{3\sin\theta\cos\theta}$$

$$= -\frac{1}{3\sin^3\theta\cos\theta} - \frac{8}{3}\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

$$= -\frac{1}{3\sin^3\theta\cos\theta} - \frac{8}{3}\cot 2\theta.$$

(15.) $\dfrac{du}{d\theta} = \tan^4\theta = \tan^2\theta\,(1 + \tan^2\theta) - \tan^2\theta$

$$= \tan^2\theta\,(1 + \tan^2\theta) - (1 + \tan^2\theta) + 1,$$

$$\therefore\ u = \frac{\tan^3\theta}{3} - \tan\theta + \theta.$$

(16.) $\dfrac{du}{d\theta} = \dfrac{1}{\tan^5\theta} = \dfrac{1 + \tan^2\theta}{\tan^5\theta} - \dfrac{1}{\tan^3\theta}$

$$= \frac{1 + \tan^2\theta}{\tan^5\theta} - \frac{1 + \tan^2\theta}{\tan^3\theta} + \frac{1}{\tan\theta},$$

$$\therefore\ u = \frac{-1}{4\tan^4\theta} + \frac{1}{2\tan^2\theta} + \text{h. l. } \sin\theta.$$

(17.) $\dfrac{du}{d\theta} = \theta^3\cos\theta \qquad p = \theta^3 \qquad \dfrac{dq}{d\theta} = \cos\theta$

$$\frac{dp}{d\theta} = 3\theta^2 \qquad q = \sin\theta,$$

$$\therefore\ u = \theta^3\sin\theta - 3\int\theta^2\sin\theta\,d\theta$$

$$p = \theta^2 \qquad \frac{dq}{d\theta} = \sin\theta$$

$$\frac{dp}{d\theta} = 2\theta \qquad q = -\cos\theta$$

$$\int\theta^2\sin\theta\,d\theta = -\theta^2\cos\theta + 2\int\theta\cos\theta\,d\theta$$

$$p = \theta \qquad \frac{dq}{d\theta} = \cos\theta$$

$$\frac{dp}{d\theta} = 1 \qquad q = \sin\theta$$

$$\int \theta \cos \theta \, d\theta = \theta \sin \theta - \int \sin \theta \, d\theta = \theta \sin \theta + \cos \theta,$$

$$\therefore \ u = \theta^3 \sin \theta + 3 \theta^2 \cos \theta - 6 \theta \sin \theta - 6 \cos \theta.$$

(18.) $\dfrac{d u}{d x} = \dfrac{x^2}{\sqrt{1 - x^2}} \sin^{-1} x.$

To integrate $\dfrac{x^2 \, dx}{\sqrt{1 - x^2}}$ $p = x$ $\dfrac{d q}{d x} = \dfrac{x}{\sqrt{1 - x^2}}$

$$\dfrac{dp}{dx} = 1 \qquad q = -\sqrt{1 - x^2}$$

$$\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = - x \sqrt{1 - x^2} + \int \sqrt{1 - x^2} \, dx$$

$$= - x \sqrt{1 - x^2} + \int \frac{dx}{\sqrt{1 - x^2}} - \int \frac{x^2 \, dx}{\sqrt{1 - x^2}},$$

$$\therefore \int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = - \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x.$$

Next, to integrate $\dfrac{x^2 \, dx}{\sqrt{1 - x^2}} \sin^{-1} x,$

$$p = \sin^{-1} x \qquad \frac{d q}{d x} = \frac{x^2}{\sqrt{1 - x^2}}$$

$$\frac{dp}{dx} = \frac{1}{\sqrt{1 - x^2}} \qquad q = - \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x,$$

$$\therefore \ u = \frac{1}{2} (\sin^{-1} x)^2 - \frac{x}{2} \sqrt{1 - x^2} \sin^{-1} x + \frac{1}{2} \int x \, dx$$

$$- \frac{1}{2} \int \frac{\sin^{-1} x \, dx}{\sqrt{1 - x^2}}$$

$$= \frac{1}{2}(\sin^{-1}x)^2 - \frac{x\sqrt{1-x^2}}{2}\sin^{-1}x + \frac{x^2}{4} - \frac{1}{4}(\sin^{-1}x)^2$$

$$= \frac{1}{4}(\sin^{-1}x)^2 - \frac{x\sqrt{1-x^2}}{2}\sin^{-1}x + \frac{x^2}{4}.$$

(19.) $\dfrac{du}{dx} = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}\sin^{-1}x.$

$$p = \sin^{-1}x \qquad \frac{dq}{dx} = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{dp}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad q = \frac{1}{(1-x^2)^{\frac{1}{2}}},$$

$$\therefore u = \frac{\sin^{-1}x}{\sqrt{1-x^2}} - \int \frac{dx}{1-x^2}$$

$$= \frac{\sin^{-1}x}{\sqrt{1-x^2}} - \frac{1}{2}\left\{\int \frac{dx}{1-x} + \int \frac{dx}{1+x}\right\}$$

$$= \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \log\frac{\sqrt{1-x}}{\sqrt{1+x}}.$$

(20.) $\dfrac{du}{dx} = \dfrac{x^2}{1+x^2}\tan^{-1}x.$

$$p = \tan^{-1}x \qquad \frac{dq}{dx} = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

$$\frac{dp}{dx} = \frac{1}{1+x^2} \qquad q = x - \tan^{-1}x,$$

$$\therefore u = \tan^{-1}x\,(x - \tan^{-1}x) - \int \frac{x\,dx}{1+x^2} + \int \frac{\tan^{-1}x\,dx}{1+x^2}.$$

$$= \tan^{-1} x \, (x - \tan^{-1} x) - \log \sqrt{1 + x^2} + \frac{(\tan^{-1} x)^2}{2}$$

$$= x \tan^{-1} x - \frac{1}{2} (\tan^{-1} x)^2 - \log \sqrt{1 + x^2}.$$

(21.) $\dfrac{du}{dx} = e^{ax} \sin^2 x.$

$$p = \sin^2 x \qquad \frac{dq}{dx} = e^{ax}$$

$$\frac{dp}{dx} = 2 \sin x \cos x \quad q = \frac{e^{ax}}{a}$$

$$u = \frac{1}{a} e^{ax} \sin^2 x - \frac{2}{a} \int e^{ax} \sin x \cos x \, dx$$

$$p = \sin x \cos x \qquad \frac{dq}{dx} = e^{ax}$$

$$\frac{dp}{dx} = \cos^2 x - \sin^2 x \qquad q = \frac{e^{ax}}{a}$$

$$= 1 - 2 \sin^2 x,$$

$$\therefore \int e^{ax} \sin x \cos x \, dx$$

$$= \frac{1}{a} e^{ax} \sin x \cos x - \frac{1}{a} \int e^{ax} \, dx + \frac{2}{a} \int e^{ax} \sin^2 x \, dx,$$

$$\therefore u = \frac{1}{a} e^{ax} \sin^2 x - \frac{2}{a^2} e^{ax} \sin x \cos x + \frac{2}{a^2} \frac{e^{ax}}{a} - \frac{4 u}{a^2},$$

$$\therefore u \left(1 + \frac{4}{a^2} \right) = \frac{e^{ax} \sin x}{a^2} (a \sin x - 2 \cos x) + \frac{2 e^{ax}}{a^3},$$

$$\therefore u = \frac{e^{ax} \sin x \, (a \sin x - 2 \cos x)}{a^2 + 4} + \frac{2 e^{ax}}{a (a^2 + 4)}.$$

(22.) $\dfrac{du}{dx} = \dfrac{1}{(a + b \cos x)^2}.$

Assume

$$\int \frac{dx}{(a + b \cos x)^2} = \frac{A \sin x}{a + b \cos x} + B \int \frac{dx}{a + b \cos x}.$$

Taking the differential coefficients

$$\frac{1}{(a + b \cos x)^2} = \frac{A \cos x \, (a + b \cos x) + b \, A \sin^2 x}{(a + b \cos x)^2}$$

$$+ \frac{B \, (a + b \cos x)}{(a + b \cos x)^2},$$

$$\therefore \; 1 = A \cos x \, (a + b \cos x) + b \, A \sin^2 x + B \, (a + b \cos x)$$

$$= A \, a \cos x + A \, b \cos^2 x + A \, b \sin^2 x + B \, a + B \, b \cos x$$

$$= (A \, a + B \, b) \cos x + A \, b + B \, a.$$

Equating like powers of $(\cos x)$,

$$A \, a + B \, b = 0, \qquad \therefore \; A = -\frac{b}{a} \, B$$

$$A \, b + B \, a = 1$$

$$B \left(-\frac{b^2}{a} + a \right) = 1, \qquad \therefore \; B = \frac{a}{a^2 - b^2},$$

$$\therefore \; A = -\frac{b}{a^2 - b^2},$$

$$\therefore \int \frac{dx}{(a + b \cos x)^2}$$

$$= -\frac{b}{a^2 - b^2} \frac{\sin x}{a + b \cos x} + \frac{a}{a^2 - b^2} \int \frac{dx}{a + b \cos x}$$

$$= \frac{1}{a^2 - b^2} \left\{ \frac{-b \sin x}{a + b \cos x} + a \int \frac{dx}{a + b \cos x} \right\}.$$

(23.) To integrate $e^{ax} \cos k\,x$.

$$\text{Let } p = \cos k\,x \qquad dq = e^{ax}\,dx,$$

$$dp = -\,k \sin k\,x\,dx \qquad q = \frac{e^{ax}}{a},$$

(1) $\therefore \displaystyle\int e^{ax} \cos k\,x\,dx = \frac{e^{ax} \cos k\,x}{a} + \frac{k}{a} \int e^{ax} \sin k\,x\,dx.$

(2) $\displaystyle\int e^{ax} \sin k\,x\,dx = \frac{e^{ax} \sin k\,x}{a} - \frac{k}{a} \int e^{ax} \cos k\,x\,dx.$

Multiplying equation (2) by $\dfrac{k}{a}$ and substituting in equation (1)

$$\int e^{ax} \cos k\,x\,dx \cdot \left(1 + \frac{k^2}{a^2} \right) = \frac{e^{ax} \cos k\,x}{a} + \frac{k\,e^{ax} \sin k\,x}{a^2},$$

$$\int e^{ax} \cos k\,x\,dx = \frac{e^{ax} (a \cos k\,x + k \sin k\,x)}{a^2 + k^2}.$$

(24.) To integrate $e^{-ax} \sin k\,x$,

$$p = \sin k\,x \qquad dq = e^{-ax}\,dx,$$

$$dp = k \cos k\,x \qquad q = -\,\frac{e^{-ax}}{a},$$

(1) $\therefore \displaystyle\int e^{-ax} \sin k\,x\,dx = -\,\frac{e^{-ax} \sin k\,x}{a} + \frac{k}{a} \int e^{-ax} \cos k\,x\,dx.$

(2) $\displaystyle\int e^{-ax} \cos k\,x\,dx$

$$= -\,\frac{e^{-ax} \cos k\,x}{a} - \frac{k}{a} \int e^{-ax} \sin k\,x\,dx.$$

Multiplying equation (2) by $\left(\dfrac{k}{a}\right)$ and substituting in equation (1)

$$\int e^{-ax} \sin kx \, dx = -\frac{e^{-ax} \sin kx}{a} - \frac{k e^{-ax} \cos kx}{a^2}$$

$$-\frac{k^2}{a^2} \int e^{ax} \sin kx \, dx$$

$$\frac{a^2 + k^2}{a^2} \int e^{-ax} \sin kx \, dx = -\frac{e^{-ax}(a \sin kx + k \cos kx}{a^2}$$

$$\int e^{-ax} \sin kx \, dx = -\frac{e^{-ax}(a \sin kx + k \cos kx)}{a^2 + k^2}.$$

$$(25.) \int \frac{d\theta}{a \cos^2 \theta + b \sin^2 \theta} = \int \frac{\dfrac{d\theta}{\cos^2 \theta}}{a + b \tan^2 \theta}$$

$$= \int \frac{d(\tan \theta)}{a + b \tan^2 \theta} = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\tan \theta \sqrt{\frac{b}{a}}\right).$$

$$(26.) \int \frac{\cos \theta \, d\theta}{(1 - e^2 \cos^2 \theta)^{\frac{3}{2}}} = \int \frac{\cos \theta \, d\theta}{(1 - e^2 + e^2 \sin^2 \theta)^{\frac{3}{2}}}$$

$$= \int \frac{\dfrac{\cos \theta}{\sin^3 \theta} \, d\theta}{(1 - e^2)^{\frac{3}{2}}\left(\dfrac{1}{\sin^2 \theta} + \dfrac{e^2}{1 - e^2}\right)^{\frac{3}{2}}}$$

$$= -\int \frac{\dfrac{1}{2} d\left(\dfrac{1}{\sin^2 \theta}\right)}{(1 - e^2)^{\frac{3}{2}}\left(\dfrac{1}{\sin^2 \theta} + \dfrac{e^2}{1 - e^2}\right)^{\frac{3}{2}}}$$

$$= \cfrac{1}{(1-e^2)^{\frac{3}{2}} \sqrt{\cfrac{1}{\sin^2 \theta} + \cfrac{e^2}{1-e^2}}}$$

$$= \cfrac{\sin \theta}{(1 - e^2) \sqrt{1 - e^2 + e^2 \sin^2 \theta}}$$

$$= \cfrac{\sin \theta}{(1-e^2)\sqrt{1 - e^2 \cos^2 \theta}}$$

(27.) $\displaystyle\int (\cos 2\theta)^{\frac{3}{2}} \cos \theta \, d\theta = \int (1 - 2 \sin^2 \theta)^{\frac{3}{2}} \cos \theta \, d\theta.$

Let $2 \sin^2 \theta = x^2 \qquad d\theta = \dfrac{dx}{\sqrt{2} \cos \theta}$

$$du = \frac{(1 - x^2)^{\frac{3}{2}} \, dx}{\sqrt{2}} = \frac{(1 - 2x^2 + x^4) \, dx}{\sqrt{2} \sqrt{1 - x^2}}$$

$$\sqrt{2} \, du = \frac{dx}{\sqrt{1 - x^2}} - \frac{2 x^2 \, dx}{\sqrt{1 - x^2}} + \frac{x^4 \, dx}{\sqrt{1 - x^2}}$$

$$\int \frac{x^2}{\sqrt{1 - x^2}} \, dx = - \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-} x,$$

$$\int \frac{x^4}{\sqrt{1 - x^2}} \, dx = - \frac{x^3}{4} \sqrt{1 - x^2} + \frac{3}{4} \int \frac{x^2}{\sqrt{1 - x^2}} \, dx$$

$$= - \sqrt{1 - x^2} \left(\frac{x^3}{4} + \frac{3 x}{2 \cdot 4} \right) + \frac{3}{8} \sin^{-1} x,$$

$$\therefore \quad \sqrt{2} \, u = \sin^{-1} x + x \sqrt{1 - x^2} - \sin^{-1} x$$

$$- \sqrt{1 - x^2} \left(\frac{x^3}{4} + \frac{3 x}{8} \right) + \frac{3}{8} \sin^{-1} x$$

$$= \sqrt{1-x^2} \left(\frac{5x}{8} - \frac{x^3}{4} \right) + \frac{3}{8} \sin^{-1} x$$

$$\sqrt{2}\,u = \sqrt{\cos 2\theta} \left(\frac{5\sqrt{2}\sin\theta}{8} - \frac{2\sqrt{2}\sin^3\theta}{4} \right)$$

$$+ \frac{3}{8} \sin^{-1} (\sqrt{2}\sin\theta),$$

$$u = \sqrt{\cos 2\theta}\, \frac{\sin\theta}{8} (5 - 4 + 4\cos^2\theta)$$

$$+ \frac{3}{8\sqrt{2}} \sin^{-1} (\sqrt{2}\sin\theta)$$

$$= \frac{\sin\theta}{8} (3 + 2\cos 2\theta) \sqrt{\cos 2\theta} + \frac{3}{8\sqrt{2}} \sin^{-1} (\sqrt{2}\sin\theta).$$

(28.) $$\int \frac{\sin\theta\, d\theta}{\sqrt{\sin^2 a - \sin^2\theta}} = \int \frac{\sin\theta\, d\theta}{\sqrt{\cos^2\theta - \cos^2 a}}$$

$$= -\int \frac{d(\cos\theta)}{\sqrt{\cos^2\theta - \cos^2 a}} = -\log \left(\sqrt{\cos^2\theta - \cos^2 a} + \cos\theta \right)$$

$$= -\log \left(\sqrt{\sin^2 a - \sin^2\theta} + \cos\theta \right) + C.$$

If $\theta = a$, $u = -\log(\cos a) + C$,

$$\therefore \int_a^\theta \frac{\sin\theta\, d\theta}{\sqrt{\sin^2 a - \sin^2\theta}} = \log \frac{\cos a}{\sqrt{\sin^2 a - \sin^2\theta} + \cos\theta}.$$

CHAPTER VI.

ON DEFINITE INTEGRALS.

In the process of differentiation all constant quantities which are merely added to, or subtracted from, those quantities which contain the variable disappear; and, on the contrary, after integration, there may be a constant quantity connected with the integral which we have not in that operation obtained. The letter C is therefore added to every integral to represent this quantity, and in order to determine its value we must in the first place find what particular value of the variable makes the integral O; we thus get two equations, both of which contain C, and between these two equations C may be eliminated. For instance, if we have $\int 3x^2 d4$ $= x^3 + $ C for the *general value* of the integral, and the problem indicates that for the *particular value* $x = a$, the integral becomes O, then O $= a^3 + $ C; by subtracting this from the general value of the integral we get $x^3 - a^3$, in which the constant C has disappeared; this latter is called the *corrected integral*, and is written thus,

$$\int_a^x 3x^2 dx = x^3 - a^3.$$

In this expression the value of the integral commences when $x = a$, and if we give another value to x, say $x = b$, then we have fully determined the value of the integral, which is now written

$$\int_a^b 3x^2 \, dx = b^3 - a^3.$$

This is called the *definite integral*, and is said to be taken between the limits $x = b$ and $x = a$: the former is called the superior limit, and the latter the inferior limit, and the operation is called integration between limits.

In general $\int f(x)\,dx = \phi(x) + C.$

In order to determine the definite integral, we must, according to the nature of the problem proposed, assign the proper limits, which we shall represent by a and b, as before; then we have

$$\int_a^b f(x)\,dx = \phi(a) - \phi(b).$$

As every function of x may represent the ordinates of a curve whose abscissa is x, it follows that the operation of integrating between limits may be applied to finding the areas and lengths of curves, the volumes and surfaces of solids of revolution, &c.

Examples.—Areas of Curves, Volumes of Solids, &c.

(1.) The general equation to a parabola of any order is $y^{m+n} = a x m^n$. Then, since $A = \int y\,dx$ taken between the proper limits, we have in this case

$$A = \int a^{\frac{m}{m+n}}\ x^{\frac{n}{m+n}}\,dx$$

$$= \frac{m+n}{m+2n}\ a^{\frac{m}{m+n}}\ x^{\frac{m+2n}{m+n}} + C.$$

Then we perceive that the area $= 0$ when $x = 0$, $\therefore$ $C = 0$; and taking the above between the limits $x = 0$ and $x = x$, since the value of the integral commences when $x = 0$ and ends when $x = x$, we have

$$\int_0^x a^{\frac{m}{m+n}}\ x^{\frac{n}{m+n}}\,dx = \frac{m+n}{m+2n}\ a^{\frac{m}{m+n}}\ x^{\frac{m+2n}{m+n}}.$$

(2.) The general equation to hyperbolas referred to their asymptotes is

$$x^m y^n = a^{m+n},$$

$$\therefore \ y^n = \frac{a^{m+n}}{x^m}, \ \text{ and } \ y = \frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}},$$

$$\therefore \ \text{A} = \int y \, dx = \int \frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}} \, dx = \int a^{\frac{m+n}{n}} \, x^{-\frac{m}{n}} \, dx$$

$$= \frac{n}{n-m} \, a^{\frac{m+n}{n}} \, x^{\frac{n-m}{n}} + \text{C}.$$

We must here determine the constant C, as in the above example, that is, find when the value of x makes the area $= 0$, &c. We must here observe, that this formula fails when $m = n$; for then A $=$ C $+$ $a^2 \log x$, which cannot be determined by the above method.

(3.) The equation to the tractrix is

$$\frac{dy}{dx} = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}},$$

$$\therefore \ y \, dx = -\, dy \, (a^2 - y^2)^{\frac{1}{2}},$$

and A $= \int y \, dx = -\int dy \, (a^2 - y^2)^{\frac{1}{2}}$

$$= -\frac{y}{2} (a^2 - y^2)^{\frac{1}{2}} - \frac{a^2}{2} \sin^{-1} \frac{y}{a} + \text{C}.$$

Then in order to find the area included by the positive axes, let $y = -\, a$, observing that C $= 0$,

$$\therefore \quad \text{whole area} = \frac{\pi a^2}{4}, \quad \because \sin^{-1} 1 = \frac{\pi}{2}.$$

Since it is shown in all works on the Differential Calculus that

$$\frac{d\,V}{dx} = \pi y^2 \text{ and } \frac{d\,S}{dx} = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

it follows that to find the volumes and surfaces of solids we have to integrate these functions between the proper limits.

Examples.

(1.) To find the volume and surface of a sphere.

The equation to the circle referred to the centre is $y^2 = r^2 - x^2$, where r represents the radius of the sphere; and as one value of r lies wholly above and the other wholly below the axis of x, we must integrate between the limits $x = -r$ and $x = r$; we have

$$V = \pi \int_{-r}^{+r} (r^2 - x^2)\, dx = \frac{4\pi}{3} r^3.$$

If we integrated this without reference to limits, the expression we should have would give the volume of the segment of a sphere; and we observe that $C = 0$ when $x = 0$, since the integral becomes 0.

Also, $$S = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\pi \int y \sqrt{1 + \frac{x^2}{y^2}}$$

$$= 2\pi \int \sqrt{y^2 + x^2}\, dx$$

$$= 2\pi \int_{-r}^{+r} (r^2)^{\frac{1}{2}}\, dx = 2\pi \times 2r^2 = 4\pi r^2.$$

We might form the integral

$$2\pi \int \sqrt{(y^2 + x^2)}\, dx = 2\pi \int \sqrt{r^2}\, dx$$

by writing the value of y^2 given in the equation to the curve.

Hence the integral $= 2\pi r x$, which is the surface of the segment, whose height is x.

(2.) To find the volume and surface of a prolate spheroid formed by the revolution of an ellipse about its major diameter.

The equation to the ellipse is $y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$, where a and b represent the major and minor semi-axes respectively.

Hence, $\mathrm{V} = \pi \int y^2\, dx = \pi \dfrac{b^2}{a^2} \int (a^2 - x^2)\, dx$

$$= \pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right) + \mathrm{C},$$

which is the volume of a spheroidal segment, remarking that $\mathrm{C} = 0$. Next we must integrate between the limits $x = -a$ and $x = a$; then we have

$$\mathrm{V} = \pi \frac{b^2}{a^2} \int_{-a}^{+a} (a^2 - x^2)\, dx = \frac{4}{3}\pi b^2 a,$$

$$\frac{d\mathrm{S}}{dx} = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and $\dfrac{dy}{dx} = -\dfrac{b}{a}\dfrac{x}{(a^2 - x^2)^{\frac{1}{2}}}$, $\therefore 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{b^2 x^2}{a^2(a^2 - x^2)}.$

If we write e^2 for $\dfrac{a^2 - b^2}{a^2}$ we have

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2 - e^2 x^2}{a^2 - x^2},$$

$$\therefore \ \mathrm{S} = 2\pi \int y \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} dx = \frac{2\pi b e}{a} \int \left(\frac{a^2}{e^2} - x^2 \right)^{\frac{1}{2}} dx$$

$$= \frac{\pi b e}{a} \left\{ x \left(\frac{a^2}{e^2} - x^2 \right)^{\frac{1}{2}} + \frac{a^2}{e^2} \sin^{-1} \frac{e x}{a} \right\} + \mathrm{C}.$$

Here $\mathrm{C} = 0$, and if we integrate between the limits $x = -a$ and $x = a$ we have the whole surface

$$= \frac{2\pi a b}{e} \left\{ e \left(1 - e^2 \right)^{\frac{1}{2}} + \sin^{-1} e \right\}.$$

(3.) To find the volume and surface of an oblate spheroid.

In order to determine the equation to this, we merely have to change a into b, and we have

$$y^2 = \frac{a^2}{b^2} \left(b^2 - x^2 \right),$$

and $\mathrm{V} = \dfrac{\pi a^2}{b^2} \displaystyle\int \left(b^2 - x^2 \right) dx = \dfrac{\pi a^2 x}{b^2} \left(b^2 - \dfrac{x^2}{3} \right) + \mathrm{C}.$

Integrating between the limits $x = -b$ and $x = +b$, observing that $\mathrm{C} = 0$; for $\mathrm{V} = 0$ when $x = 0$, $\therefore$ the whole volume

$$= \frac{\pi a^2}{b^2} \int_{-b}^{+b} \left(b^2 - x^2 \right) dx = \frac{4}{3} \pi a^2 b,$$

and the surface of the oblate sphere may be found in the same manner as the last.

(4.) To find the volume of a circular spindle.

Here let O be the centre of the circle of which ACB is a segment, and let OC = the rad. = r be perpendicular to AB, OD = a; GD = x and the chord AB = $2c$; GF = y; (see figure page 103.) then we have

$$r^2 = x^2 + (y + a)^2,$$

$$\therefore \ y^2 = r^2 - a^2 - x^2 - 2ay,$$

$$\mathbf{V} = \pi \int y^2 dx = \pi \int (r^2 - a^2 - x^2 - 2ay)\, dx$$

$$= \pi \left\{ (r^2 - a^2)\, x + \frac{x^3}{3} - 2a \int y\, dx \right\}$$

$$= \pi \left\{ (r^2 - a^2)\, x - \frac{x^3}{3} - 2a \times \text{gen. area DGFC} \right\} + \text{C}.$$

Here C = 0, and the above gives the volume of the frustum HECE, the double of which is the whole frustum HFIL. The limits are $x = -c$, and $x = +c$, we have volume of the spindle

$$= 2\pi \left\{ \tfrac{1}{3} c^3 - a \times \text{gen. area ACB} \right\}.$$

(5.) To find the volume of an elliptic spindle.

Let ACB be the generating arc of the ellipse, in which $\mathbf{AD} = c$, $\mathbf{OD} = i$, $\mathbf{MO} = a = $ semi-axis major, and $\mathbf{OC} = b = $ semi-axis minor; $\mathbf{DG} = x$, and $\mathbf{FG} = y$. Then from the property of the ellipse c being the
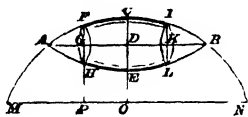
$$a : b :: (a^2 - x^2)^{\frac{1}{2}} : \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} = \mathbf{PF}.$$

Hence $y = \dfrac{b(a^2 - x^2)^{\frac{1}{2}}}{a} - i,$

$$\therefore \ y^2 = \frac{b^2}{a^2}(a^2 - x^2) - \frac{2bi\sqrt{a^2 - x^2}}{a} + i^2,$$

$$\therefore \ \mathbf{V} = \pi \int y^2 dx$$

$$= \pi \int \left(b^2 - \frac{b^2 x^2}{a^2} - \frac{2bi}{a}\sqrt{a^2 - x^2} + i^2 \right) dx$$

$$= \pi \int \left\{ b^2 - \frac{b^2 x^2}{a^2} - i^2 - 2i \cdot \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} + 2i^2 \right\} dx$$

$$= \pi \int \left\{ b^2 \cdot \frac{c^2 - x^2}{a^2} - 2iy \right\} dx$$

$$= \pi \left\{ b^2 x \cdot \frac{3c^2 - x^2}{3a^2} - 2i \cdot \text{area DGFC} \right\} + \text{C}$$

$$= \text{volume of ECFH, and when } x = c, \ \text{C} = o,$$
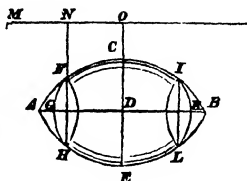
$\therefore$ we have $\pi \left\{ b^2 c \cdot \dfrac{3c^2 - c^2}{3a^2} - 2i \cdot \text{area DAC} \right\}$,

or, $\pi \left\{ \dfrac{2b^2}{3a^2} \cdot c^3 - 2i \cdot \text{area DAC} \right\}$,

the double of which will give the volume of the whole spindle.

(6.) To find the volume of a hy-perbolic spindle.

Putting $i =$ the central distance O D, $c =$ A B, and retaining the notation employed in the last, we have by the nature of the curve,

$$\text{OM} : \text{OC} = b :: \sqrt{(a^2 + x^2)} : \text{NF} = \frac{b \sqrt{a^2 + x^2}}{a},$$

$$\therefore y = \text{OD} - \text{NF} = i - \frac{b \sqrt{a^2 + x^2}}{a}.$$

$$\therefore \text{V} = \int y^2 \, dx$$

$$= \pi \int \left\{ i^2 - \frac{2bi}{a} \sqrt{a^2 + x^2} + b^2 \left(\frac{a^2 + x^2}{a^2} \right) \right\} dx$$

$$= \pi \int \left\{ b^2 - i^2 + \frac{b^2 x^2}{a^2} + \left(i - \frac{(a^2 + x^2)^{\frac{1}{2}}}{a} \right) 2i \right\} dx$$

$$= \pi \int \left\{ b^2 - i^2 + \frac{b^2 x^2}{a^2} + 2iy \right\} dx$$

$$= \pi \left\{ 2i \times \text{area GFCD} - \frac{b^2 x}{a^2} \left(\frac{1}{4} c^2 - \frac{x^2}{3} \right) \right\} + \text{C}$$

$$= \pi \left\{ 2i \times \text{area GFCD} - \frac{b^2 x}{a^2} \times \left(\frac{3c^2 - 4x^2}{12} \right) \right\} + \text{C},$$

and $C = 0$. This is the volume of the frustum FCEH, and for the volume of the spindle we have

$$2\pi \left\{ 2i \times \text{area AFCD} - \frac{b^2 c^3}{12 a^2} \right\}.$$

(7.) To find the surface of a circular spindle.

Retaining the notation used in the last, we have
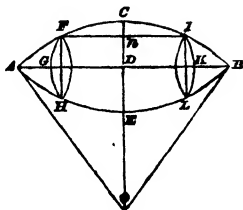
$$no = y + a$$

$$= \sqrt{OF^2 - Fn^2} = \sqrt{r^2 - x^2},$$

$$\therefore y = (r^2 - x^2)^{\frac{1}{2}} - a,$$

and $\dfrac{dy}{dx} = -\dfrac{x}{(r^2 - x^2)^{\frac{1}{2}}}$ and $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{x^2}{r^2 - x^2}$,

$$\therefore S = 2\pi \int y \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int \left\{ (r^2 - x^2)^{\frac{1}{2}} - a \right\} \left\{ \sqrt{\frac{r^2}{r^2 - x^2}} \right\} dx$$

$$= 2\pi r \int \left\{ 1 - \frac{a}{\sqrt{r^2 - x^2}} \right\} dx = 2\pi r \left\{ x - a \sin^{-1} \frac{x}{r} \right\} + C,$$

and $C = 0$, the surface of the part HECF; and if $x = c$, we have

$$S = 2\pi r \left\{ c - a \sin^{-1} \frac{c}{r} \right\},$$

the double of which will be the surface of the whole spindle ACBE
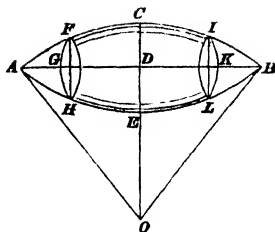
(8.) To find the volume of a parabolic spindle.

Put $CD = h$, $AB = 2c$, and $x = AG$, and $y = FG$.

From the property of the parabola

$$AD^2 : AG . GB :: CD : EG,$$

$$c^2 : x(2c - x) :: h : y,$$

$$\therefore y = \frac{h(2cx - x^2)}{c^2}, \therefore y^2 = \frac{h^2}{c^4}(4c^2x^2 - 4cx^3 + x^4),$$

$$\therefore V = \pi \int y^2 dx$$

$$= \pi \frac{h^2}{c^4} \int (4c^2x^2 - 4cx^3 + x^4) dx = \frac{\pi h^2}{c^4}\left(\frac{4c^2x^3}{3} - cx^4 + \frac{x^5}{5}\right),$$

the volume of AFH, since $C = 0$; when $x = c$, we have

$$\frac{\pi h^2}{c^4}\left(\frac{4c^5}{3} - c^5 + \frac{c^5}{5}\right) = \frac{8}{15}\pi h^2 c =$$
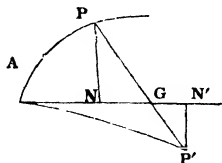
volume of the semi-spindle; and, as we have found the volume of the part AFH, if we subtract this from the whole semi-spindle, we shall have the frustum EHCF, the double of which will be volume of the whole frustum EHIL.

In the same manner as in the last example, we may find the surface of the parabolic spindle.

(9.) In a parabola find the area included between the curve, its evolute, and its radius of curvature.

area parabola $ANP = \int y \, dx$

$$= 2\sqrt{a} \int \sqrt{x} \, dx = \frac{4}{3}\sqrt{a} \, x^{\frac{3}{2}},$$

area evolute $AN'P' = \int \beta \, d\alpha = \frac{2}{3\sqrt{3a}} \int (\alpha - 2a)^{\frac{3}{2}} d\alpha$

$$= \frac{4}{15 \sqrt{8a}} (a - 2a)^{\frac{4}{2}} = \frac{4}{15 \sqrt{3a}} 9 \sqrt{3} x^{\frac{3}{2}} = \frac{12 x^{\frac{3}{2}}}{5 \sqrt{a}},$$

$$\text{subnormal NG} = y \frac{dy}{dx} = y \frac{2a}{y} = 2a,$$

$$\text{area PNG} = \frac{\text{NP . NG}}{2} = ay = 2a^{\frac{3}{2}} x^{\frac{1}{2}}$$

$$\text{GN}' = \text{AN}' - \text{AG} = 3x + 2a - x - 2a = 2x,$$

$$\text{area P'N'G} = \frac{\text{GN}' \cdot \text{N'P}'}{2} = \beta x = \frac{y^3}{4a^2} x$$

$$= \frac{4ax^2}{4a^2} 2 \sqrt{ax} = \frac{2x^{\frac{5}{2}}}{\sqrt{a}},$$

$$\text{area APP}' = \text{APN} + \text{NPG} + \text{AN'P}' - \text{GN'P}'$$

$$= \frac{4}{3} \sqrt{a} x^{\frac{3}{2}} + 2a^{\frac{3}{2}} x^{\frac{1}{2}} + \frac{12 x^{\frac{3}{2}}}{5 \sqrt{a}} - \frac{2x^{\frac{5}{2}}}{\sqrt{a}}$$

$$= \frac{20 a x^{\frac{3}{2}} + 30 a^2 x^{\frac{1}{2}} + 6 x^{\frac{5}{2}}}{15 \sqrt{a}}$$

$$= \frac{2 \sqrt{x}}{\sqrt{a}} \left(a^2 + \frac{2}{3} ax + \frac{1}{5} x^2 \right).$$

(10.) To find the length of the spiral of Archimedes.

$$r = a \qquad \frac{dr}{d\theta} = a,$$

$$\frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr} = \frac{1}{a} \sqrt{r^2 + a^2}$$

$$s = \frac{1}{a} \int \sqrt{r^2 + a^2}\, dr = \frac{1}{a} \int \frac{r^2\, dr}{\sqrt{r^2 + a^2}} + a \int \frac{dr}{\sqrt{r^2 + a^2}}$$

$$= \frac{r}{a} \sqrt{r^2 + a^2} - \frac{1}{a} \int dr \sqrt{r^2 + a^2} + a \int \frac{dr}{\sqrt{r^2 + a^2}}$$

$$= \frac{r}{2a} \sqrt{r^2 + a^2} - \frac{a}{2} \int \frac{dr}{\sqrt{r^2 + a^2}} + a \int \frac{dr}{\sqrt{r^2 + a^2}}$$

$$= \frac{r}{2a} \sqrt{r^2 + a^2} + \frac{a}{2} \int \frac{dr}{\sqrt{r^2 + a^2}}$$

$$= \frac{r}{2a} \sqrt{r^2 + a^2} + \frac{a}{2} \log\left(\frac{\sqrt{r^2 + a^2} + r}{a} \right).$$

(11.) On A B, the diameter of a given semicircle A C B, take A D = the chord A C; join C, D; bisect C D in P, and find the equation and area of the locus of P.

Join A, P, and put A P = y, $\angle$ PAD = ϕ, and AB = $2a$; then A C = A D = $2a \cos 2\phi$, and A P or $y = 2a \cos \phi \cos 2\phi$. Let A P = y be drawn in an opposite direction; then ϕ will become $180° + \phi$, and A P or y will become = $2a \cos(180° = \phi) \times \cos(360° + 2\phi) = -2a \cos \phi \cos 2\phi = -y$ which indicates that when ϕ passes $180°$, the point P will begin at B, and describe exactly the same curve which it has described; hence, when ϕ arrives at $180°$, the curve is complete. When $y = 0$, ϕ has three values from $0°$ to $180°$, viz., $45°$, $90°$, and $135°$, which shows that the point P returns thrice to the point A, and therefore describes three noduses; 2 area ABP = $\int y^2 d\phi = 4a^2 \int d\phi \cos^2 \phi (1 - 4 \sin^2 \phi \cos^2 \phi) = a^2 \phi + \frac{1}{3} a^2 \sin \phi (3 \cos \phi - 2 \cos^3 \phi + 8 \cos^5 \phi)$, this between $\phi = 0°$, and $\phi = 45°$, gives $\frac{1}{4}$ circle $+ \frac{2}{3} a^2$, or $1·45206\, a^2$ for the area A P B E A, and between $\phi = 45°$, and $\phi = 90°$, gives $\frac{1}{4}$ circle $- \frac{2}{3} a^2$, or $·118733'\, a^2$ for the area of the other two small and equal noduses, each of which are therefore = $\frac{1}{8}$ circle $- \frac{1}{3} a^2$, or $·05936'\, a^2$; hence the area of the entire curve comprehending the three noduses is equal to the semicircle A C B.
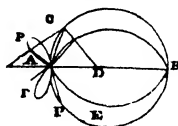
(12.) A C D B is a given circle, whose diameter is A B ; on the chord A D, which is an arithmetical mean between the chord A C and the diameter A B, take A P, a geometrical mean between A C and A B, and show the nature and area of the curve which is the locus of P.
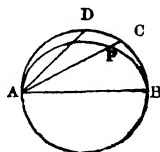
Put A B = a, A P = r, and ∠ P A B = θ; then A D = a cos θ, and A C = 2 A D — A B = 2 a cos θ — a, and consequently A P² or r² = A C . A B = a² (2 cos θ — 1), the polar equation to the curve. For the quadrature we have

$$2 \text{ area ABP} = \int r^2 \, d\theta = a^2 \int d\theta \, (2 \cos \theta - 1)$$
$$= 2\,a^2 \sin \theta - a^2 \, \theta.$$

To ascertain the limit, put r = 0; then r² or a² (2 cos θ — 1) = 0, and cos θ = ½ ; therefore the curve makes an angle of 60° with the diameter A B; hence, taking the above between

θ = 0, and θ = 60°, we get $a^2 \sqrt{3} - \frac{1}{6} a^2 . 2\pi$, or $a^2 \times \cdot 6848.$

(13.) Let the given circle C E D, whose diameter is C D, touch the indefinite right line A D in D, and from the given point A, draw the right line A E C, on which take A P = the sine of the arc E C, and find the equation and area of the curve which is the locus of P.
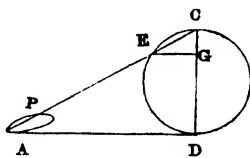
The lines being drawn as enunciated, put C D = a, ∠ D A C = φ ; then A D = a cot φ, and, by similar △s, C D : A D :: C G : E G = cot φ × C G; but, by the circle,

$$E G = (a . C G - C G^2)^{\frac{1}{2}}; \text{ whence } C G = \frac{a}{\cot^2 \varphi + 1} = a$$

sin² φ, E G = A P = (a² sin² φ — a² sin⁴ φ)^½ = ½ a sin 2φ; the polar equation of the curve. When φ = 90°, A P = 0, when φ = 45°, A P = ½ a, and when φ = 0°, A P = 0; hence A is a *punctum duplex*.

Quadrature.—$\int \frac{1}{2} \mathrm{A\,P} \times d\varphi = \frac{1}{8} a^2 \int d\varphi \sin^2 2\varphi = \frac{1}{8} a^2$ $(-\frac{1}{8} \sin 4\varphi + \frac{1}{2}\varphi)$, which, between $\varphi = 90°$, and $\varphi = 0°$, gives $\frac{1}{8}$ of the given circle for the area of the *nodus* A P A.

(12.) Let A B, the base of the right-angled triangle A B C, be produced till B P be equal to the diameter of the inscribed circle. Required the equation, quadrature, and greatest possible ordinate of the curve which is the locus of P, the hypothenuse A C being given.

Put the hypothenuse A C $= a$, $3\cdot14159265 = \pi$, and angle C A B $= x$; then, by trigonometry, A B $= a \cos x$, and B C $= a \sin x$; whence B P $=$ A B $+$ B C $-$ A C $= a (\cos x + \sin x - 1)$, and A P $= a$ $(2 \cos x + \sin x - 1)$, the polar equation of the curve. When $x = 90°$, A P $= 0$, and when $x = 0°$, A P $= a$; $\therefore$ the curve commences at A and terminates at C. The maximum polar ordinate is when $\cos x = 2 \sin x$, or when $\sin x = \frac{1}{5} \sqrt{5}$, or $x = 26° \ 33' \ 9''$.
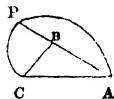
Quadrature.—The differential of the area is $= \frac{1}{2} \mathrm{A\,P}^2 \, dx$ $= \frac{1}{2} a^2 \, dx \times (4 \cos^2 x + 4 \cos x \sin x + \sin^2 x - 4 \cos x - 2 \sin x + 1) = \frac{1}{2} a^2 \, dx \times \left(\frac{7}{2} - 4 \cos x - 2 \sin x + \frac{3}{2} \cos 2x + 2 \sin 2 x \right)$, and the integrals give the area $= \frac{1}{2} a^2 \times \left(\frac{7 x}{2} - 4 \sin x + 2 \cos x + \frac{3}{4} \sin 2 x - \cos 2 x \right)$; which, between $x = 0°$, and $x = 90°$, gives $\frac{7}{8} a^2 \pi - 2 a^2$ for the area of the whole curve A P C, or in numbers $= \cdot74889357 \, a^2$, as required.

The area of the space inclosed by the curve

$$a y^2 = x^2 \sqrt{a^2 - x^2} \text{ is } \frac{8}{5} a^2.$$

$$\text{Area} = \int y \, dx = \frac{1}{\sqrt{a}} \int x \, (a^2 - x^2)^{\frac{1}{4}} \, dx$$

$$= - \frac{1}{\sqrt{a}} \cdot \frac{2}{5} (a^2 - x^2)^{\frac{5}{4}} + \mathrm{C}$$

If $x = 0$ Area $= \dfrac{2}{5} a^2 + $ C.

If $x = a$ Area $= 0$,

$\therefore$ Whole area $= \dfrac{4 \cdot 2}{5} a^3$.

The length of the curve, $y \log \left(\dfrac{e^x + 1}{e^x - 1} \right)$

from $x = 1$ to $x = 2$ is $= \log (e + e^{-1})$.

$$y = \log \left(\frac{e^x + 1}{e^x - 1} \right),$$

$$\frac{dy}{dx} = \frac{e^x (e^x - 1) - e^x (e^x + 1)}{(e^x - 1)^2} \cdot \frac{e^x - 1}{e^x + 1}$$

$$= \frac{-2 e^x}{e^{2x} - 1},$$

$$1 + \frac{dy^2}{dx^2} = \frac{e^{4x} - 2 e^{2x} + 1 + 4 e^{2x}}{(e^{2x} - 1)^2} = \left(\frac{e^{2x} + 1}{e^{2x} - 1} \right)^2,$$

$$\therefore \text{ s} = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \int \frac{e^{2x} + 1}{e^{2x} - 1} \frac{dx}{}$$

$$= \int \frac{e^x \, dx}{e^x + 1} + \int \frac{dx}{e^x - 1} = \int \frac{e^x \, dx}{e^x + 1} + \int \frac{e^x \, dx}{e^x - 1} - \int dx$$

$$\text{s} = \log (e^{2x} - 1) - x + \text{C}.$$

If $x = 2$ s $= \log (e^4 - 1) - 2 + $ C.

$x = 1$ s $= \log (e^2 - 1) - 1 + $ C,

$\therefore$ from $x = 2$ to $x = 1$ $S = \log(e^4 - 1) - \log(e^2 - 1) - 1,$

$$S = \log(e^2 + 1) - \log e$$

$$= \log\left(\frac{e^2 + 1}{e}\right) = \log(e + e^{-1}).$$

(14.)   The length of the curve $8\,a^3 y = x^4 + 6\,a^2 x^2$, measured from the origin of co-ordinates, is

$$\frac{x}{8a^3}(x^2 + 4a^2)^{\frac{3}{2}}$$

$$y = \frac{x}{8\,a^3}(x^4 + 6\,a^2 x^2)$$

$$\frac{dy}{dx} = \frac{1}{8\,a^3}(4x^3 + 12\,a^2 x) = \frac{1}{2\,a^3}(x^2 + 3\,a^2 x)$$

$$1 + \frac{dy^2}{dx^2} = \frac{1}{4\,a^6}(x^6 + 6\,a^2 x^4 + 9\,a^4 x^2 + 4\,a^6)$$

$$= \frac{1}{4\,a^6}\{x^6 + 3\,a^2 x^4 + 3\,a^4 x^2 + a^6 + 3\,a^2(x^4 + 2\,a^2 x^2 + a^4)\}$$

$$= \frac{1}{4\,a^6}\left\{\left(x^2 + a^2\right)^3 + 3\,a^2(x^2 + a^2)^2\right\}$$

$$= \frac{(x^2 + a^2)^2}{4\,a^6}(x^2 + 4a^2),$$

$$\therefore s = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{1}{2\,a^3}\int(x^2 + a^2)\sqrt{x^2 + 4\,a^2}\,dx$$

$$= \frac{1}{2\,a^3}\frac{x}{4}(x^2 + 4a^2)^{\frac{3}{2}}.$$

(15.) The volume generated by the curve $y^2 (x - 4a) = ax (x - 3a)$ revolving about the axis of x, from $x = 0$ to $x = 3a$ is $= \frac{1}{2} \pi a^3 (15 - 16 \log 2)$.

$$y^2 = a \frac{x^2 - 3ax}{x - 4a} = a \left(x + a + \frac{4a^2}{x - 4a} \right),$$

$$\therefore V = \pi \int y^2 \, dx = \pi a \left(\frac{x^2}{2} + ax + 4a^2 \log (x - 4a) \right) + C.$$

If $x = 0$ $\qquad 0 = 4\pi a^3 \log (- 4a) + C.$

If $x = 3a$

$$0 = \pi a \left(\frac{9a^2}{2} + 3a^2 + 4a^2 \log (- a) \right) + C,$$

$$\therefore V = \pi a \left\{ \frac{15a^2}{2} - 4a^2 \left\{ \log (- 4a) - \log (- a) \right\} \right\}$$

$$= \frac{\pi}{2} a^3 \left\{ 15 - 8 \log 4 \right\}$$

$$= \frac{\pi}{2} a^3 \left\{ 15 - 16 \log 2 \right\}.$$

(16.) Find the area of the curve in which

$$r = \frac{(a^2 - b^2) \sin \theta \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

$$\text{area} = \frac{1}{2} \int r^2 \, d\theta = \frac{1}{2} \int \frac{(a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta$$

$$= \frac{1}{2} \int \frac{(a^2 - b^2)^2 \sin^2 \theta \, d\theta}{a^2 \tan^2 \theta + b^2}.$$

Let $x = \tan \theta,$ $d\theta = \dfrac{1}{1 + x^2} \, dx$ $\sin \theta = \dfrac{x}{\sqrt{1 + x^2}}$

$$\text{area} = \frac{1}{2} \int \frac{(a^2 - b^2)^2 \dfrac{x^2}{1 + x^2} \dfrac{1}{1 + x^2} \, dx}{a^2 x^2 + b^2}$$

$$= \frac{1}{2} \int \frac{(a^2 - b^2)^2 x^2 \, dx}{(a^2 x^2 + b^2)(1 + x^2)^2}.$$

Let $\dfrac{U}{V} = \dfrac{A}{a^2 x^2 + b^2} + \dfrac{B}{(1 + x^2)^2} + \dfrac{C}{1 + x^2}$

$$(a^2 - b^2)^2 x^2$$

$$= A(1 + x^2)^2 + B(a^2 x^2 + b^2) + c(1 + x^2)(a^2 x^2 + b^2).$$

Let $x^2 = -\dfrac{b^2}{a^2},$ $\therefore - (a^2 - b^2)^2 \dfrac{b^2}{a^2} = A \left(\dfrac{a^2 - b^2}{a^2} \right)^2$

$$A = - a^2 b^2.$$

Let $x = \sqrt{-1}$; $-(a^2 - b^2)^2 = - B(a^2 - b^2),$

$$\therefore B = a^2 - b^2,$$

$$\therefore (a^2 - b^2)^2 x^2 - (a^2 - b^2)(a^2 x^2 + b^2) + a^2 b^2 (1 + x^2)^2$$

$$= C(1 + x^2)(a^2 x^2 + b^2)$$

$$- (a^2 - b^2)(1 + x^2) b^2 + a^2 b^2 (1 + x^2)^2 = C(1 + x^2)$$

$$(a^2 x^2 + b^2)$$

$$b^2 (1 + x^2)(a^2 + a^2 x^2 - a^2 + b^2) = C(1 + x^2)(a^2 x^2 + b^2)$$

$$C = b^2,$$

$$\therefore \int \frac{(a^2 - b^2)\, x^2\, d x}{(a^2\, x^2 + b^2)\, (1 + x^2)^2}$$

$$= -a^2\, b^2 \int \frac{d x}{a^2\, x^2 + b^2} + (a^2 - b^2) \int \frac{d x}{(1 + x^2)^2}$$

$$+ b^2 \int \frac{d x}{1 + x^2}.$$

But $\int \dfrac{d x}{(1 + x^2)^2} = \int \dfrac{d x}{1 + x^2} - \int \dfrac{x^2\, d x}{(1 + x^2)^2}$

$$= \int \frac{d x}{1 + x^2} + \frac{x}{2\,(1 + x^2)} - \frac{1}{2} \int \frac{d x}{1 + x^2}$$

$$= \frac{1}{2} \int \frac{d x}{1 + x^2} + \frac{x}{2\,(1 + x^2)},$$

$$\therefore \text{area} = - \frac{a^2\, b^2}{2} \int \frac{d x}{a^2\, x^2 + b^2} + \frac{a^2 + b^2}{4} \int \frac{d x}{1 + x^2}$$

$$+ \frac{x}{4\,(1 + x^2)}$$

$$= - \frac{d b}{2} \tan^{-1} \frac{a x}{b} + \frac{a^2 + b^2}{4} \tan^{-1} x + \frac{x}{4\,(1 + x^2)}.$$

Taking this between limits $\theta = 0$ and $\theta = 90$;

or, $x = 0 \quad x = \infty$

$$\text{area} = \frac{a^2 + b^2}{4} \frac{\pi}{2} - \frac{a b}{2} \frac{\pi}{2} = \frac{(a - b)^2}{4} \frac{\pi}{2}.$$

(17.) The length of the epicycloid after one revolution of the generating circle $= 8\dfrac{b}{a}(a+b)$, and the area between the epicycloid and the circle $= \pi b^2\left(3 + \dfrac{2\,b}{a}\right)$

$p^2 = c^2\left(\dfrac{r^2 - a^2}{c^2 - a^2}\right)$, equation to epicycloid in terms of rad. vector and perp. on tangent where $c = a + 2\,b$.

$$\frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}$$

$$r^2 - p^2 = \frac{c^2 r^2 - a^2 r^2 - c^2 r^2 + a^2 c^2}{c^2 - a^2} = \frac{a^2(c^2 - r^2)}{c^2 - a^2}$$

$$\frac{ds}{dr} = \frac{\sqrt{c^2 - a^2}}{a}\frac{r}{\sqrt{c^2 - r^2}}$$

$$s = \pm\frac{\sqrt{c^2 - a^2}}{a}\sqrt{c^2 - r^2} + C.$$

If $r = a + 2b = c$,   then $s = 0$, $C = 0$.

If $r = a$,   then $s = \dfrac{c^2 - a^2}{a} = \dfrac{a^2 + 4ab + 4b^2 - a^2}{a}$

$$s = \pm\frac{4b}{a}(a+b),$$

hence whole length of arc of epicycloid

$$= \frac{8b}{a}(a+b).$$

Also to find area $\dfrac{d\theta}{dr} = \dfrac{p}{r\sqrt{r^2 - p^2}}$

$$r^2\,d\theta = \frac{pr\,dr}{\sqrt{r^2 - p^2}} = \frac{cr\sqrt{r^2 - a^2}}{\sqrt{c^2 - a^2}} \times \frac{\sqrt{c^2 - a^2}\,dr}{a\sqrt{c^2 - r^2}}$$

$$= \frac{c}{a}\,\frac{r\sqrt{r^2 - a^2}\,dr}{\sqrt{c^2 - r^2}}$$

$$\text{area} = \frac{1}{2}\int r^2\,d\theta = \frac{c}{2a}\int \frac{r\,dr\sqrt{r^2 - a^2}}{\sqrt{c^2 - r^2}}.$$

Let $r^2 - a^2 = z^2$, $r\,dr = z\,dz$, $c^2 - r^2 = c^2 - a^2 - z^2$,

$$\text{area} = \frac{c}{2a}\int \frac{z^2\,dz}{\sqrt{\beta^2 - z^2}}; \quad \text{if } c^2 - a^2 = \beta^2$$

$$= -\frac{c}{2a}\,z\,\sqrt{\beta^2 - z^2} + \frac{c}{2a}\int dz\,\sqrt{\beta^2 - z^2}$$

$$= -\frac{c}{2a}\,z\,\sqrt{\beta^2 - z^2} + \frac{c\beta^2}{2a}\int \frac{dz}{\sqrt{\beta^2 - z^2}}$$

$$-\frac{c}{2a}\int \frac{z^2\,dz}{\sqrt{\beta^2 - z^2}},$$

$$\text{area} = -\frac{c}{4a}\,z\,\sqrt{\beta^2 - z^2} + \frac{c\beta^2}{4a}\sin^{-1}\frac{z}{\beta} + \text{C}.$$

If $r = a$ then $z = 0$, and area $= 0$,

$r = c$ then $\beta^2 - z^2 = 0$ $z^2 = c^2 - a^2$,

$$\therefore \text{ semi-area} = \frac{c\,(c^2 - a^2)}{4\,a} \sin^{-1} \frac{\sqrt{c^2 - a^2}}{\sqrt{c^2 - a^2}}$$

$$= \frac{c\,(c^2 - a^2)}{4\,a}\,\frac{\pi}{2}$$

$$\text{area circle} = \frac{a}{2}\,2\,\pi\,b = \pi\,a\,b,$$

$$\text{area between epicycloid and circle} = \frac{c\,(c^2 - a^2)}{4\,a}\,\pi - \pi\,a\,b$$

$$= \frac{(a + 2\,b)\,4\,b\,(a + b)}{4\,a}\,\pi - \pi\,a\,b$$

$$= \frac{\pi}{a}\,(a^2 b + 3\,a\,b + 2\,b^3 - a^2 b)$$

$$= \pi\,b^2\left(3 + \frac{2\,b}{a}\right).$$

(18.) Find the length of the curve where

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$s = \int \sqrt{1 + \frac{dy^2}{dx^2}} = \int \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}$$

$$= \int \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} = \int \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{3}{2}\,a^{\frac{1}{3}}\,x^{\frac{2}{3}}$$

Taking it between the limits $x = 0$, $x = a$,

$$s = \frac{3}{2}\,a.$$

The whole length of the curve $4 \times \frac{3}{2}\,a = 6\,a.$

(19.) If $h = $ height of a parabolic frustum, a and b the radii of the ends, show that

$$\text{Frustum} = \frac{\pi h}{2}(a^2 + b^2).$$

Equation to parabola $y^2 = 4mx$.

$$\mathbf{V} = \pi \int_x^{x+h} y^2\, dx = 4\pi m \int_x^{x+h} x\, dx$$

$$= 2\pi m \left\{ (x+h)^2 - x^2 \right\} = 2\pi m (2xh + h^2)$$

$$= \frac{\pi h}{2} \left\{ 4mx + 4m(x+h) \right\} = \frac{\pi h}{2}(a^2 + b^2).$$

(20.) Find the area of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right).$$

$$\text{Area} = \int y\, dx = \frac{a}{2} \int \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) dx$$

$$= \frac{a}{2}\left(a\, e^{\frac{x}{a}} - a\, e^{-\frac{x}{a}} \right),$$

$$= \frac{a}{2} \sqrt{ a^2 e^{\frac{2x}{a}} + 2a^2 + a^2 e^{-\frac{2x}{a}} - 4a^2 }$$

$$= \frac{a}{2} \sqrt{ 4y^2 - 4a^2 } = a \sqrt{ y^2 - a^2 }.$$

(21.) Find the area of $x^4 y^4 - a^4 y^4 = a^8$,

$$y = \frac{a^2}{\sqrt[4]{x^4 - a^4}},$$

$$\text{area} = \int y \, dx = a^2 \int \frac{dx}{\sqrt[4]{x^4 - a^4}} = a^2 \int \frac{dx}{x \, z},$$

If $x^4 - a^4 = x^4 z^4,$

$$x = \frac{a}{\sqrt[4]{1 - z^4}},$$

$$\log x = \log a - \frac{1}{4} \log (1 - z^4),$$

$$\frac{dx}{x} = \frac{z^3}{1 - z^4} \, dz,$$

$$\therefore \text{area} = a^2 \int \frac{1}{z} \frac{z^3}{1 - z^4} \, dz = a^2 \int \frac{z^2}{1 - z^4} \, dz$$

$$= \frac{a^2}{2} \left\{ \int \frac{1}{1 - z^2} - \int \frac{1}{1 + z^2} \right\}$$

$$= \frac{a^2}{4} \int \frac{1}{1 + z} + \frac{a^2}{4} \int \frac{1}{1 - z} - \frac{a^2}{2} \int \frac{1}{1 + z^2}$$

$$= \frac{a^2}{4} \log \frac{1 + z}{1 - z} - \frac{a^2}{2} \tan^{-1} z.$$

But $z^4 = \frac{x^4 - a^4}{x^4} = \frac{a^8}{x^4 y^4},$

$$\therefore \text{area} = \frac{a^2}{4} \log \frac{1 + \dfrac{a^2}{xy}}{1 - \dfrac{a^2}{xy}} - \frac{a^2}{2} \tan^{-1} \frac{a^2}{xy}$$

$$= \frac{a^2}{2} \left\{ \log \sqrt{\frac{xy + a^2}{xy - a^2}} - \tan^{-1} \frac{a^2}{xy} \right\}.$$

(22.) Find the volume generated by the revolution of the witch round its asymptote.

$$y^2 = 4a^2\,\frac{2a - x}{x} \quad \text{equation to witch,}$$

$$xy^2 = 8a^3 - 4a^2x,$$

$$x = \frac{8a^3}{y^2 + 4a^2},$$

$$\text{Volume} = \pi\int x^2\,dy = 16a^4\,\pi\int\frac{4a^2\,dy}{(y^2 + 4a^2)^2}$$

$$= 16a^4\pi\int\frac{dy}{y^2 + 4a^2} - 16a^4\,\pi\int\frac{y^2\,dy}{(y^2 + 4a^2)^2}.$$

$$\text{But}\int y\,\frac{y\,dy}{(y^2 + 4a^2)^2} = -\frac{1}{2}\,\frac{y}{y^2 + 4a^2} + \frac{1}{2}\int\frac{dy}{y^2 + 4a^2}.$$

$$\text{Volume} = 8a^4\,\pi\,\frac{y}{y^2 + 4a^2} + 8a^4\,\pi\int\frac{dy}{y^2 + 4a^2}$$

$$= 8a^4\,\pi\,\frac{y}{y^2 + 4a^2} + 4a^3\,\pi\,\tan^{-1}\frac{y}{2a}.$$

Taking between the limits of $y = \infty$ and $y = 0$.

Volume $= 4a^3\,\pi\,\tan^{-1}\infty \quad = 2a^3\,\pi^2$, and whole volume generated by curve both above and below the abscissa $= 4a^3\,\pi^2$.

MISCELLANEOUS EXAMPLES.

(1.) $\int \sqrt{2ax - x^2} \cdot dx$

$$= \int \frac{(2ax - x^2) \, dx}{\sqrt{2ax - x^2}}$$

$$= \int \frac{x \, dx \, \{a + (a - x)\}}{\sqrt{2ax - x^2}}$$

$$= \int \frac{a x \, dx}{\sqrt{2ax - x^2}} + \int \frac{(a - x) \, x \, dx}{\sqrt{2ax - x^2}}$$

$$\int \frac{(a - x) \, dx \cdot x}{\sqrt{2ax - x^2}}$$

$$= x \sqrt{2ax - x^2} - \int \sqrt{2ax - x^2} \, dx,$$

$$\int \frac{a x \, dx}{\sqrt{2ax - x^2}} = - \int \frac{\{(a - x) \, a - a^2\} \, dx}{\sqrt{2ax - x^2}}$$

$$= - a \sqrt{2ax - x^2} + a^2 \cdot \text{ver} \sin^{-1} \frac{x}{a},$$

$$\therefore 2 \int \sqrt{2ax - x^2} \cdot dx$$

$$= (x - a) \sqrt{2ax - x^2} + a^2 \, \text{ver} \sin^{-1} \frac{x}{a},$$

$$\therefore \int \sqrt{2ax - x^2} \, . \, dx$$

$$= \frac{(x-a)\sqrt{2ax - x^2}}{2} + \frac{a^2}{2} \text{ ver sin}^{-1} \frac{x}{a}.$$

This integral is used by Earnshaw, in finding the centre of gravity of the area of a cycloid.

(2.) $$\int \frac{x^2 \, dx}{\sqrt{2ax + x^2}}.$$

By formula of Reduction,

$$\int \frac{x^n \, dx}{\sqrt{2ax + x^2}}$$

$$= \frac{x^{n-1}\sqrt{2ax + x^2}}{n} - \frac{a(2n-1)}{n}\int \frac{x^{n-1} \, dx}{\sqrt{2a + x^2}}.$$

If $n = 2$, this gives $\displaystyle\int \frac{x^2 \, dx}{\sqrt{2ax + x^2}}$

$$= \frac{x\sqrt{2ax + x^2}}{2} - \frac{3a}{2}\int \frac{x \, dx}{\sqrt{2ax + x^2}}.$$

Also $\displaystyle\int \frac{x \, dx}{\sqrt{2ax + x^2}}$

$$= \sqrt{2ax + x^2} - a \int \frac{dx}{\sqrt{2ax + x^2}}.$$

G

And $\int \dfrac{d\,x}{\sqrt{2\,ax + x^2}} = \log\{x + a + \sqrt{2\,ax + x^2}\},$

$$\therefore \int \dfrac{x\,d\,x}{\sqrt{2\,ax + x^2}}$$

$$= \sqrt{2\,ax + x^2} - a \log\{a + a + \sqrt{2\,ax + x^2}\}.$$

$$\therefore \int \dfrac{x^2\,dx}{\sqrt{2\,ax + x^2}} = \dfrac{x\sqrt{2\,ax + x^2}}{2} - \dfrac{3\,a}{2}\sqrt{2\,ax + x^2}$$

$$+ \dfrac{3\,a^2}{2} \log\{x + a + \sqrt{2\,ax + x^2}\}.$$

(3.) $\qquad \displaystyle\int_a^o \dfrac{x^2\,dx}{(2\,ax - x^2)^{\frac{3}{2}}}$

let $p = x \qquad dp = dx, \qquad dq = \dfrac{x\,dx}{(2\,ax - x^2)^{\frac{3}{2}}},$

$$\therefore q = \int \dfrac{x\,dx}{(2\,ax - x)^{\frac{3}{2}}} = \int (2a - x)^{-\frac{3}{2}}\,x^{\frac{1}{2}}\,dx =$$

$$\int (2ax^{-1} - 1)^{-\frac{3}{2}}\,x^{-2}\,dx$$

$$= -\dfrac{1}{2\,a}\int (2ax^{-1} - 1)^{-\frac{3}{2}} \times -2\,a\,x^{-2}\,dx$$

$$= \dfrac{(2ax^{-1} - 1)^{-\frac{1}{2}}}{a} = \dfrac{1}{a}\sqrt{\dfrac{x}{2a - x}}.$$

$$\int p\, dq = pq - \int q\, dp$$

$$= \frac{x}{a} \sqrt{\frac{x}{2a - x}} - \frac{1}{a} \int \frac{\sqrt{x} \cdot dx}{\sqrt{2a - x}}$$

$$= \frac{x}{a} \sqrt{\frac{x}{2a - x}} - \frac{1}{a} \int \frac{x\, dx}{\sqrt{2ax - x^2}};$$

but by the formula of reduction,

$$\int \frac{x\, dx}{\sqrt{2ax - x^2}} = - \sqrt{2ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a},$$

$$\therefore \int \frac{x^2\, dx}{(2ax - x^2)^{\frac{3}{2}}}$$

$$= \frac{x}{a} \sqrt{\frac{x}{2a - x}} + \frac{1}{a} \sqrt{2ax - x^2} - \operatorname{vers}^{-1} \frac{x}{a},$$

taken between the limits of 0 and a,

$$\int_a^o \frac{x^2\, dx}{(2ax - x^2)^{\frac{3}{2}}} = 1 + 1 - \frac{\pi}{2} = 2 - \frac{\pi}{2} = \cdot 4292.$$

This integral is used in Barlow on the strength of mate·ials. See pages 364, 365.

$$(4.)\ \int \frac{x^2\, dx}{x^4 + a^4} = \left(\frac{1}{4a} \log \frac{x - a}{x + a} + \frac{1}{2a} \tan^{-1} \frac{x}{a} \right) \frac{1}{a}.$$

$$(5.)\ \int \frac{dx}{(x^2 + b)^2} = \frac{x}{2b(x^2 + b)} + \frac{1}{2b\sqrt{b}} \tan^{-1} \frac{x}{\sqrt{b}}.$$

(6.)
$$\int \frac{dx}{(x+a)(x^2+b)^2}$$

$$= \frac{1}{(a^2+b)^2} \log \frac{x+a}{\sqrt{x^2+b}} + \frac{1}{2b(a^2+b)} \cdot \frac{ax+b}{x^2+b}$$

$$+ \frac{a(a^2+3b)}{2b\sqrt{b}(a^2+b)^2} \cdot \tan^{-1} \frac{x}{\sqrt{b}},$$

(7.)
$$\int \frac{x^4\,dx}{(a+bx^2)^3}$$

$$= -\left(\frac{5x^3}{8b} + \frac{3ax}{8b^2} \right) \cdot \frac{1}{(a+bx)^2} + \frac{3}{8b^2} \int \frac{dx}{a+bx^2}$$

(8.) $\int \dfrac{x^2\,dx}{(a+bx)^2} = \dfrac{1}{b^3}(a+bx) - \dfrac{a^2}{b^3} \cdot \dfrac{1}{a+bx} - \dfrac{2a}{b^3} \log(a+bx).$

The formula of reduction

(9.)
$$\int \frac{dx}{x^m\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{(m-1)x^{m-1}}$$

$$+ \frac{m-2}{m-1} \int \frac{dx}{x^{m-2}\sqrt{1-x^2}},$$

gives the following integrals, taking m odd

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\log\left(\frac{1+\sqrt{1-x^2}}{x} \right) + c$$

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{2x^2} + \frac{1}{2} \int \frac{dx}{x\sqrt{1-x^2}}$$

$$\int \frac{dx}{x^5\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{4x^4} + \frac{3}{4} \int \frac{dx}{x^3\sqrt{1-x^2}}$$

$$\int \frac{dx}{x^7 \sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{6x^6} + \frac{5}{6} \int \frac{dx}{x^5 \sqrt{1-x^2}}.$$

Taking m even

$$\int \frac{dx}{x^2 \sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} + c$$

$$\int \frac{dx}{x^4 \sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{3x^3} + \frac{2}{3} \int \frac{dx}{x^2 \sqrt{1-x^2}},$$

&c.

The following integrals may be done by the general form

$$\int x^{n-1} (a + bx^n)^{\frac{p}{q}}.$$

(10.) $$\int \frac{dx}{x \sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \frac{a+bx-\sqrt{a}}{a+bx+\sqrt{a}}.$$

(11.) $$\int \frac{dx}{x \sqrt{4+3x}} = \frac{\sqrt{4+3x}}{4x} - \frac{3}{16} \log \frac{\sqrt{4+3x}-2}{\sqrt{4+3x}+2}.$$

(12.) $$\int \frac{dx}{x \sqrt{bx-a}} = \frac{2}{\sqrt{a}} \cdot \sin^{-1} \sqrt{\frac{bx-a}{bx}}.$$

(13.) $$\int \frac{dx}{x(a+bx)^{\frac{3}{2}}} = \frac{2}{a\sqrt{a+bx}}$$
$$+ \frac{1}{a\sqrt{a}} \log \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}.$$

(14.) $$\int \frac{x^2 dx}{\sqrt[3]{2+3x}} = (2+3x)^{\frac{2}{3}} \left\{ \frac{4-4x+5x^2}{40} \right\}.$$

(15.) $$\int \frac{dx}{(a+bx^2)^{\frac{3}{2}}} = \left(\frac{1}{3a(a+bx^2)} + \frac{2}{3a^2} \right) \frac{x}{\sqrt{a+bx^2}}.$$

(16.) $$\int \frac{x^{\frac{1}{2}} dx}{(1+x^{\frac{1}{2}})^{\frac{1}{2}}} = 4(1+x^{\frac{1}{2}})^{\frac{1}{2}} \left\{ \frac{1}{5}(1+x^{\frac{1}{2}})^2 - \frac{2}{3}(1+x^{\frac{1}{2}}) + 1 \right\}.$$

(17.) $\int \dfrac{x^{3n-1} dx}{(a+bx^n)^{\frac{1}{4}}} = \dfrac{2(a+bx^n)^{\frac{1}{2}}}{15 n b^3} (3 b^2 x^{2n} - 4abx^n + 8a^2).$

(18.) $\int \dfrac{dx}{(2ax+x^2)^{\frac{5}{2}}} = -\left\{ \dfrac{2}{3(2ax+x^2)} - \dfrac{2}{3 a^2} \right\}$

$$\dfrac{x+a}{a^2 \sqrt{2ax+x^2}}.$$

(19.) $$\int \dfrac{e^x x\, dx}{(1+x)^2} = \dfrac{e^x}{1+x}.$$

(20.) Show that $e^{\int \frac{d\theta}{\sin\theta}} = \tan\dfrac{\theta}{2}$

(21.) $\int e^{\sqrt{x}} x\, dx = 2 e^{\sqrt{x}} (x^{\frac{3}{2}} - 3x + 6\sqrt{x} - 6).$

(22.) $\int e^{ax} \sin^2 x\, dx = \dfrac{2 e^{ax}}{a(a^2+4)}$

$\qquad + \dfrac{e^{ax}.\sin x}{a^2+4} (a \sin x + 2\cos x),$

(23.) $\int e^x \cos^2 x = \dfrac{e^x}{2} + \dfrac{e^x \cos 2x}{10} + \dfrac{e^x \sin 2x}{5}.$

(24.) $\int \dfrac{dx}{\sin^2 x \cos^4 x} = 2 \tan x + \dfrac{\tan^3 x}{3} + \dfrac{1}{\tan x};$

also, show that the integral is

$$\dfrac{1}{3} \cdot \dfrac{1}{\sin x \cos^3 x} - \dfrac{8}{3} \cot 2x *.$$

(25.) $\int \dfrac{d\theta}{1 - \epsilon^2 \cos^2 \theta} = \dfrac{1}{\sqrt{1-\epsilon^2}} \tan^{-1}\left\{ \dfrac{\tan\theta}{\sqrt{1-\epsilon^2}} \right\},\ (\epsilon < 1).$

---

* It is observed by De Morgan, that we are liable, in using artifices of integration to produce results which appear different, but which in fact only differ by a constant. This discrepancy does not appear when the integrals are taken between definite limits, since $\phi a - \phi b = \phi a + C - (\phi b + C)$, are the same. See De Morgan's Differential and Integral Calculus, p. 116.

(26.) $\int \dfrac{d\theta}{1 - \iota^2 \cos^2 \theta} = \dfrac{1}{2\sqrt{1 - \iota^2}} \log \left\{ \dfrac{\sqrt{\iota^2 - 1} \cos \theta - \sin \theta}{\sqrt{\iota^2 - 1} \cos \theta + \sin \theta} \right\}$

$$(\iota > 1).$$

(27.) $\int \dfrac{d\theta}{\sin^3 (a\theta + b) \cos^2 (a\theta + b)} = \dfrac{3}{2a} \log \tan \left\{ \dfrac{a\theta + b}{2} \right\}$

$$+ \dfrac{1}{a} \left\{ \dfrac{1}{\cos (a\theta + b)} - \dfrac{\cos a\theta + b}{2(\sin^2 a\theta + b)} \right\}.$$

(28.) $\int \dfrac{d\theta}{\sec \theta \; \overline{\csc} \; \theta} = \dfrac{\text{vers } 2\theta}{4},$

(29.) $\int d\theta \csc 2\theta = \dfrac{1}{2} \log \tan \theta,$

(30.) $\int \dfrac{d\theta}{\cos^2 n\theta} = \dfrac{1}{n} \tan n\theta,$

(31.) $\int \dfrac{d\theta}{1 - \tan^2 \theta} = \theta + \dfrac{1}{8} \log \dfrac{1 + \sin 2\theta}{1 - \sin 2\theta},$

(32.) $\int \dfrac{d\theta}{\sin^4 \theta \cos \theta} = \dfrac{1}{3 \sin^3 \theta} - \dfrac{1}{\sin \theta} + \dfrac{1}{2} \log \tan \left(45 + \dfrac{\theta}{2} \right)$

(33.) $\int \dfrac{\sin^3 \theta \, d\theta}{\cos^2 \theta} = \cos \theta + \sec \theta,$

(34.) $\int \dfrac{d\theta}{\sin \theta \cos^3 \theta} = \dfrac{1}{2 \cos^2 \theta} + \log \tan \theta,$

(35.) $\int \dfrac{\sin^3 \theta \, d\theta}{\cos^8 \theta} = \dfrac{\sin^2 \theta}{5 \cos^7 \theta} - \dfrac{2}{5 \cdot 7 \cos^7 \theta}.$

(36.) $\int_r^R \int_\phi^\theta r^2 \sin \theta \, d\theta \, dr = -\dfrac{1}{3}(R^3 - r^3)(\cos \theta - \cos \theta).$

This integral is used by Professor Moseley on the arch. See the " Mechanical Principles of Engineering and Architecture," page 467.

(37.) A O B is a quadrant of a circle, O its centre; draw any chord B E D, and O E perpendicular to it, upon which take

O P equal to half the cosine of the arc B D, and determine the quadrature of the locus of P.

(38.) C P and C D are conjugate diameters of an ellipse, of which the semiaxes are a and b; P F is perpendicular to D C: it is required to find the area of the curve, which is the locus of F.

(39.) A right line drawn from a given point cuts a given circle, and the intercepted chord is the minor principal axis of an ellipse, whose area is equal to that of the given circle. Find the quadrature of the curve described by the vertex of the ellipse.

(40.) Supposing the arc of a semicircle to be stretched out into a straight line, and an indefinite number of perpendiculars erected on it, each equal to the versed line of the corresponding arc; what would be the length of the curve traced out by the tops of the perpendiculars?

(41.) The polar equation to the parabola is $r = \dfrac{a}{\cos^2 \dfrac{\theta}{2}}$; show

that the area $= a^2 \left(\tan \dfrac{\theta}{2} + \dfrac{1}{3} \tan^3 \dfrac{\theta}{2} \right)$.

(42.) The equation to the lemniscate being $(x^2 + y^2)^2 = x^2 - y^2$, find its area between the limits of $x = 0$ and $x = 1$.

(43.) Let the base A B of a right-angled plane triangle be given, and in the variable hypothenuse A C, let there be continually taken C P equal to the perpendicular C B. Find the equation and quadrature of the curve, which is the locus of the point P.

(44.) A C B is a given quadrant of a circle; A the centre, and D any point in the curve. Draw O D perpendicular to A B, and take D P = B O; then the area of the curve, which is the locus of P, will be = the circular segment C D B C.

(45.) The perpendicular B C of a right-angled triangle A B C is given, and in the variable hypothenuse A C, let A P be taken, so as always to be equal to B C; required the distance B P of nearest approach, and the quadrature of the curve, which is the locus of P.

(46.) Find the content of the solid generated by the revolution of the curve, whose equation is $(a^2 + x^2)\, y^2 - x^2 (a^2 - x^2) = 0$; about the axis of x.